Mathematical Induction

Part Two

Announcements

- Problem Set 1 due Friday, October 4 at the start of class.
- Problem Set 1 checkpoints graded, will be returned at end of lecture.
  - Afterwards, will be available in the filing cabinets in the Gates Open Area near the submissions box.

The **principle of mathematical induction** states that if for some $P(n)$ the following hold:

$P(0)$ is true

If it starts true…

and

…and it stays true…

For any $n \in \mathbb{N}$, we have $P(n) \rightarrow P(n+1)$

then

…then it's always true.

For any $n \in \mathbb{N}$, $P(n)$ is true.

Theorem: For any natural number n, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

Proof: By induction. Let $P(n)$ be

$$P(n) \equiv \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

For our base case, we need to show $P(0)$ is true, meaning that

$$\sum_{i=0}^{-1} 2^i = 2^0 - 1$$

Since $2^0 - 1 = 0$ and the left-hand side is the empty sum, $P(0)$ holds.

For the inductive step, assume that for some $n \in \mathbb{N}$, that $P(n)$ holds, so

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1$$

We need to show that $P(n + 1)$ holds, meaning that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

To see this, note that

$$\sum_{i=0}^{n} 2^i = \left(\sum_{i=0}^{n-1} 2^i\right) + 2^n = 2^n - 1 + 2^n = 2\left(2^n\right) - 1 = 2^{n+1} - 1$$

Thus $P(n + 1)$ holds, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is,
  - that $P(0)$ is true, and that
  - whenever $P(n)$ is true, $P(n + 1)$ is true,

  the proof is usually valid.

Theorem: For any natural number n, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$

Proof: By induction on n. For our base case, if $n = 0$, note that

$$\sum_{i=0}^{-1} 2^i = 0 = 2^0 - 1$$

and the theorem is true for 0.

For the inductive step, assume that for some n the theorem is true. Then we have that

$$\sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} i + 2^n = 2^n - 1 + 2^n = 2\left(2^n\right) - 1 = 2^{n+1} - 1$$

so the theorem is true for $n + 1$, completing the induction. ■

Variations on Induction: **Starting Later**

Induction Starting at 0

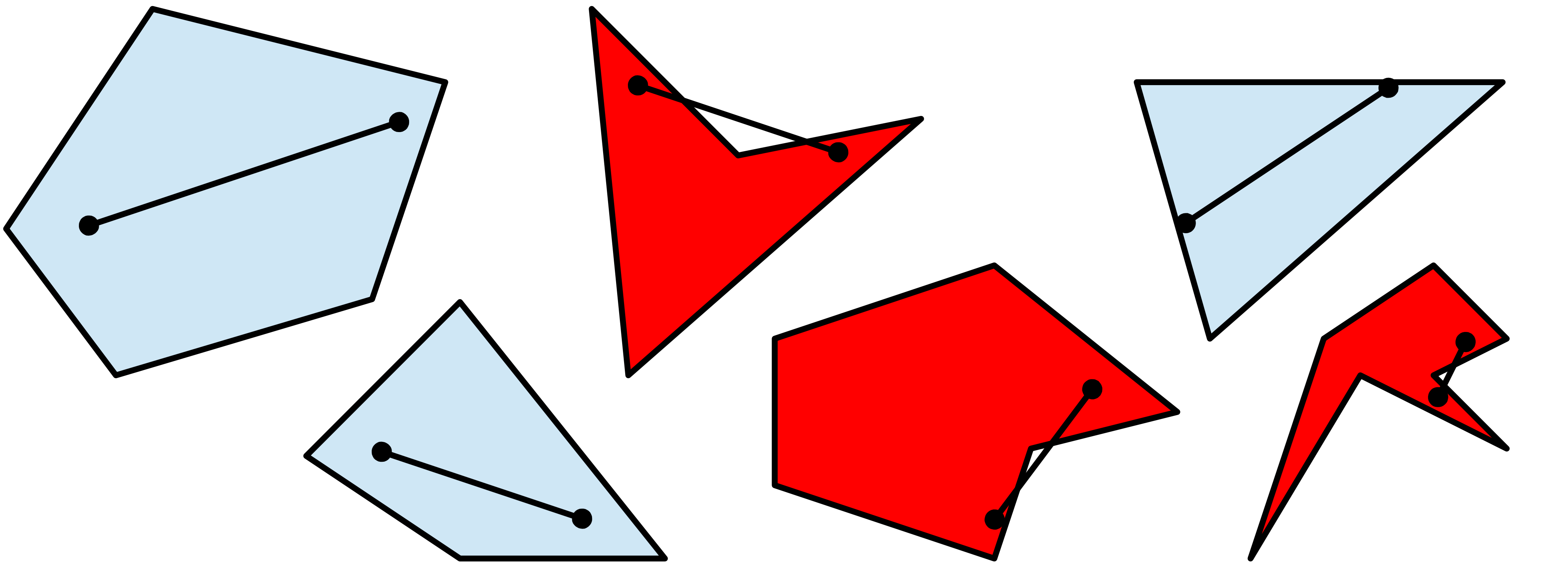
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $n \geq 0$, that $P(n) \rightarrow P(n + 1)$.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{k}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{k}$:
  - Show that $P(\boldsymbol{k})$ is true.
  - Show that for any $n \geq \boldsymbol{k}$, that $P(n) \rightarrow P(n + 1)$.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{k}$.
- Pretty much identical to before, except that the induction begins at a later point.

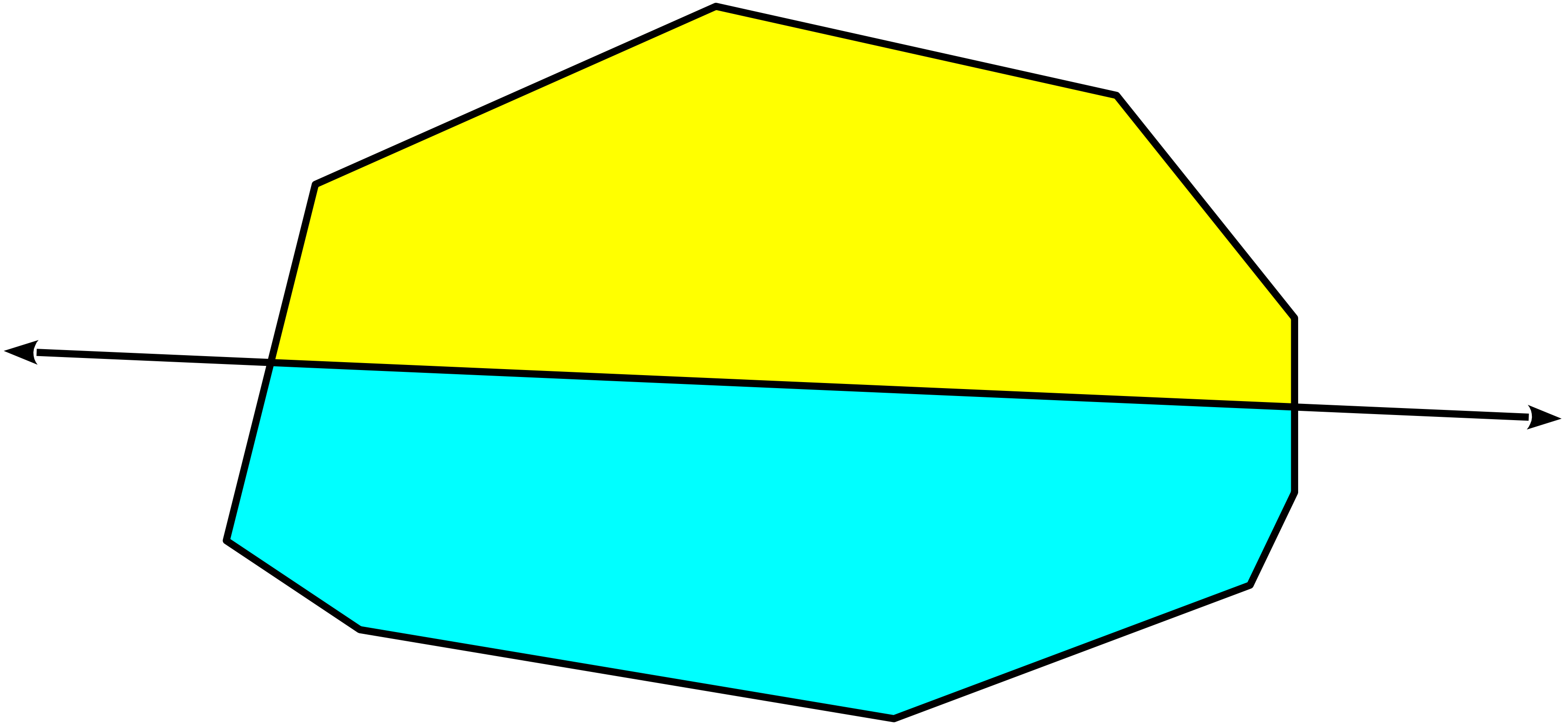
Convex Polygons

- A **convex polygon** is a polygon where, for any two points in or on the polygon, the line between those points is contained within the polygon.



Useful Fact

- **Theorem:** Any line drawn through a convex polygon splits that polygon into two convex polygons.



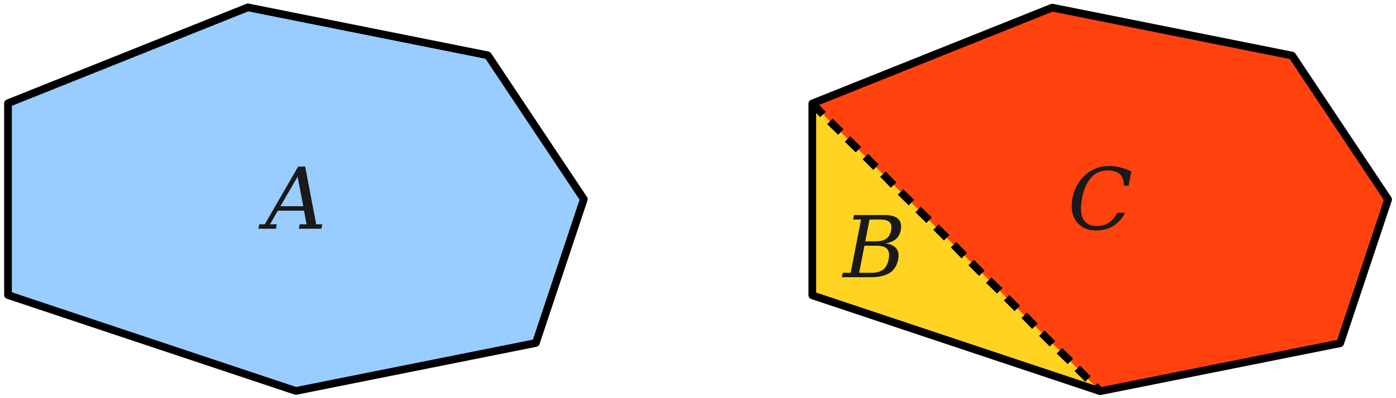
Summing Angles

- Interesting fact: the sum of the angles in a convex polygon depends only on the number of vertices in the polygon, not the shape of that polygon.
- **Theorem:** For any convex polygon with n vertices, the sum of the angles in that polygon is $(n - 2) \cdot 180°$.
  - Angles in a triangle add up to 180°.
  - Angles in a quadrilateral add up to 360°.
  - Angles in a pentagon add up to 540°.

Theorem: The sum of the angles in any convex polygon with n vertices is $(n - 2) \cdot 180°$.

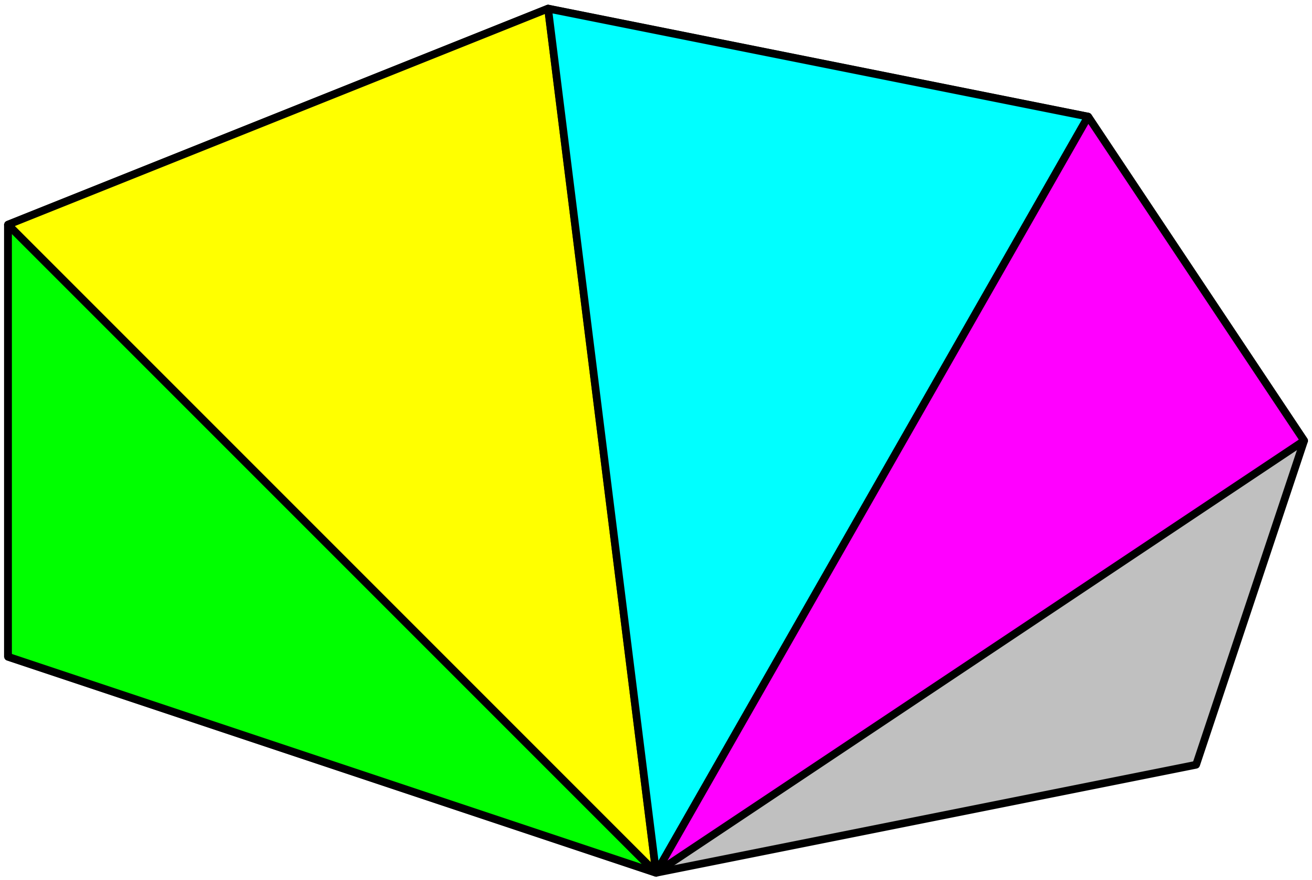
Proof: By induction. Let $P(n)$ be "all convex polygons with n vertices have angles that sum to $(n - 2) \cdot 180°$." We will prove $P(n)$ holds for all $n \in \mathbb{N}$ where $n \geq 3$. As a base case, we prove P(3): the sum of the angles in any convex polygon with three vertices is 180°. Any such polygon is a triangle, so its angles sum to 180°.

For the inductive step, assume for some $n \geq 3$ that $P(n)$ holds and all convex polygons with n vertices have angles that sum to $(n-2) \cdot 180°$. We prove $P(n+1)$, that the sum of the angles in any convex polygon with $n+1$ vertices is $(n-1) \cdot 180°$. Let A be an arbitrary convex polygon with $n+1$ vertices. Take any three consecutive vertices in A and draw a line from the first to the third, as shown here:



The sum of the angles in A is equal to the sum of the angles in triangle B (180°) and the sum of the angles in convex polygon C (which, by the IH, is $(n - 2) \cdot 180°$). Therefore, the sum of the angles in A is $(n-1) \cdot 180°$. Thus $P(n + 1)$ holds, completing the induction. ■

A Different Proof Approach

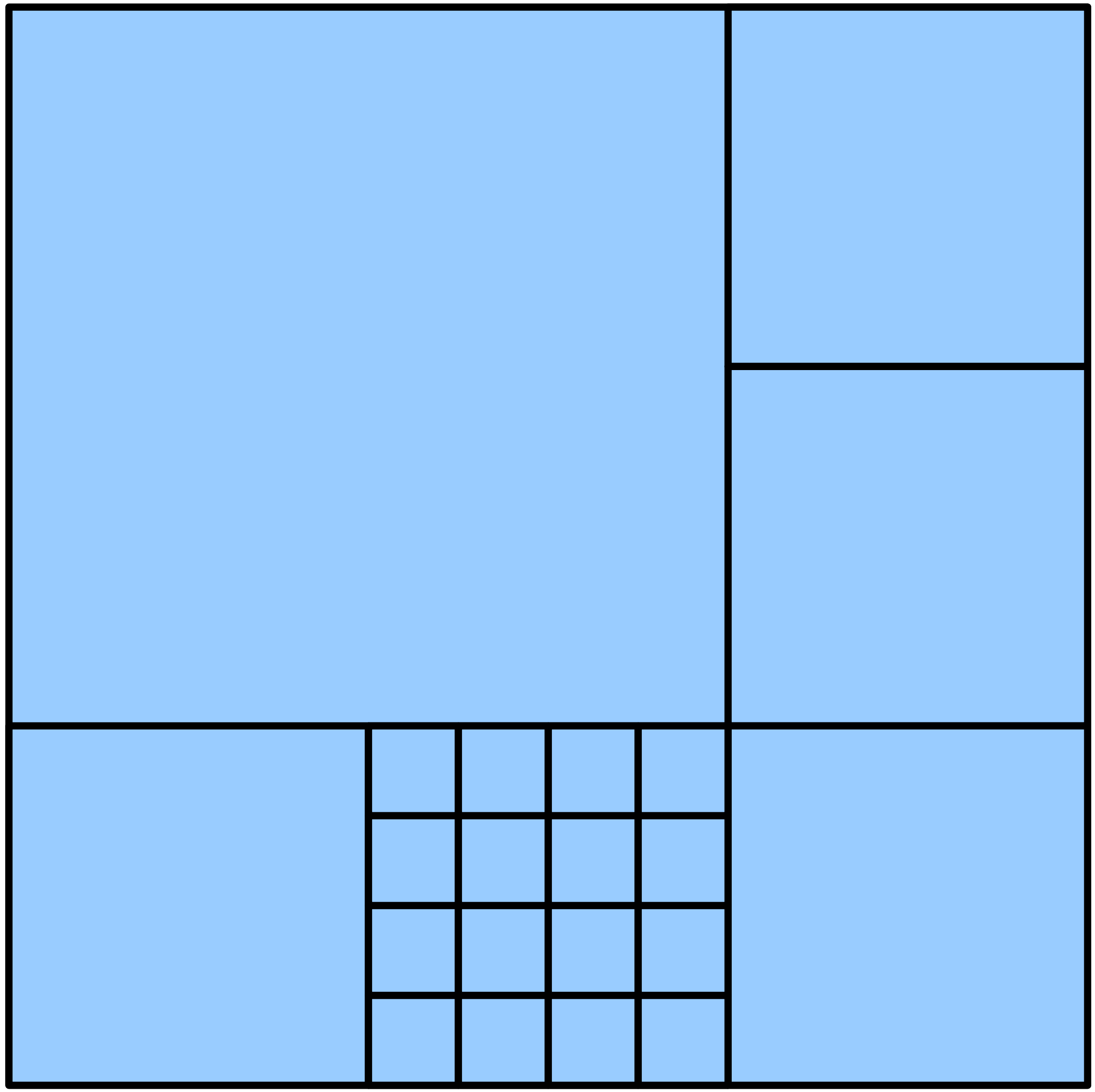


Using Induction

- Many proofs that work by induction can be written non-inductively by using similar arguments.
- Don't feel that you *have* to use induction; it's one of many tools in your proof toolbox!

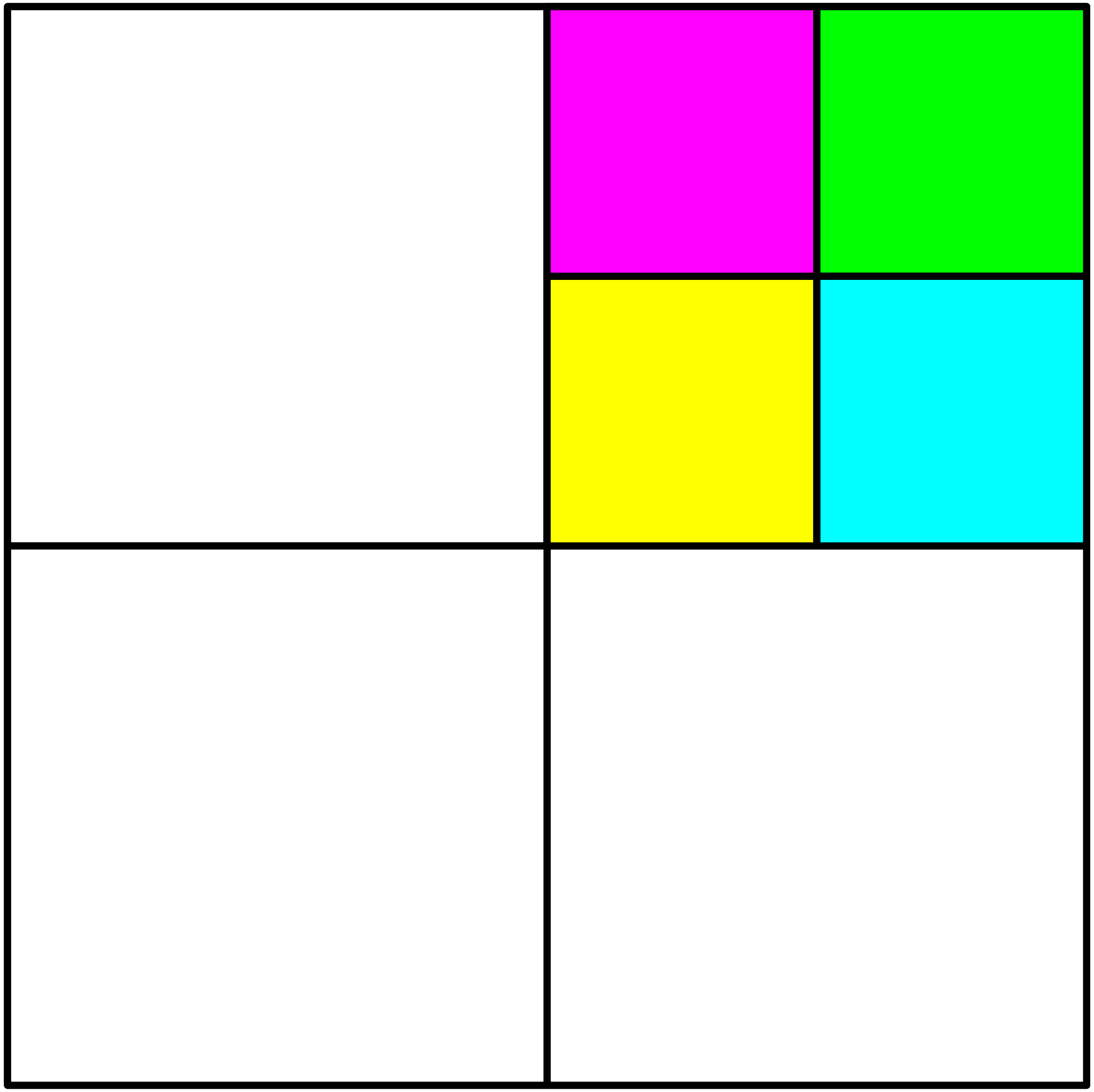
Variations on Induction: **Bigger Steps**

Subdividing a Square



For what values of n can a square be subdivided into n squares?

The Key Insight



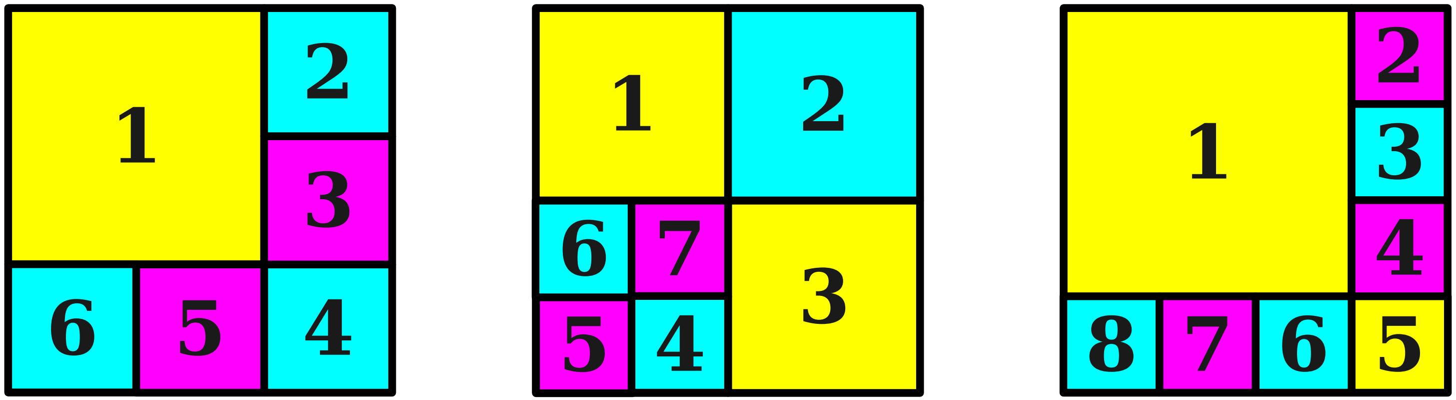
The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, it is possible to subdivide a square into n squares.

Proof: By induction. Let $P(n)$ be "a square can be subdivided into n squares." We will prove $P(n)$ holds for all $n \geq 6$.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:



For the inductive step, assume that for some $n \geq 6$ that $P(n)$ is true and a square can be subdivided into n squares. We prove $P(n + 3)$, that a square can be subdivided into $n + 3$ squares. To see this, obtain a subdivision of a square into n squares. Then, choose a square and split it into four equal squares. This removes one of the n squares and adds four more, so there are now $n + 3$ total squares. Thus $P(n + 3)$ holds, completing the induction. ■

Generalizing Induction

- When doing a proof by induction:
  - Feel free to use multiple base cases.
  - Feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!

Variations on Induction: **Complete Induction**

An Observation

$$\boldsymbol{P(n) \to P(n+1)}$$

$\boldsymbol{P(0)}$ $\boldsymbol{P(1)}$ $\boldsymbol{P(2)}$ $\boldsymbol{P(3)}$ $\boldsymbol{P(4)}$ $\boldsymbol{P(5)}$

An Observation

- In a proof by induction, the inductive step works as follows:
  - Assume that for some particular n that $P(n)$ is true.
  - Prove that $P(n + 1)$ is true.
- Notice: When trying to prove $P(n + 1)$, we already know $P(0)$, $P(1)$, $P(2)$, …, $P(n)$ but only assume $P(n)$ is true.
- Why are we discarding all the intermediary results?

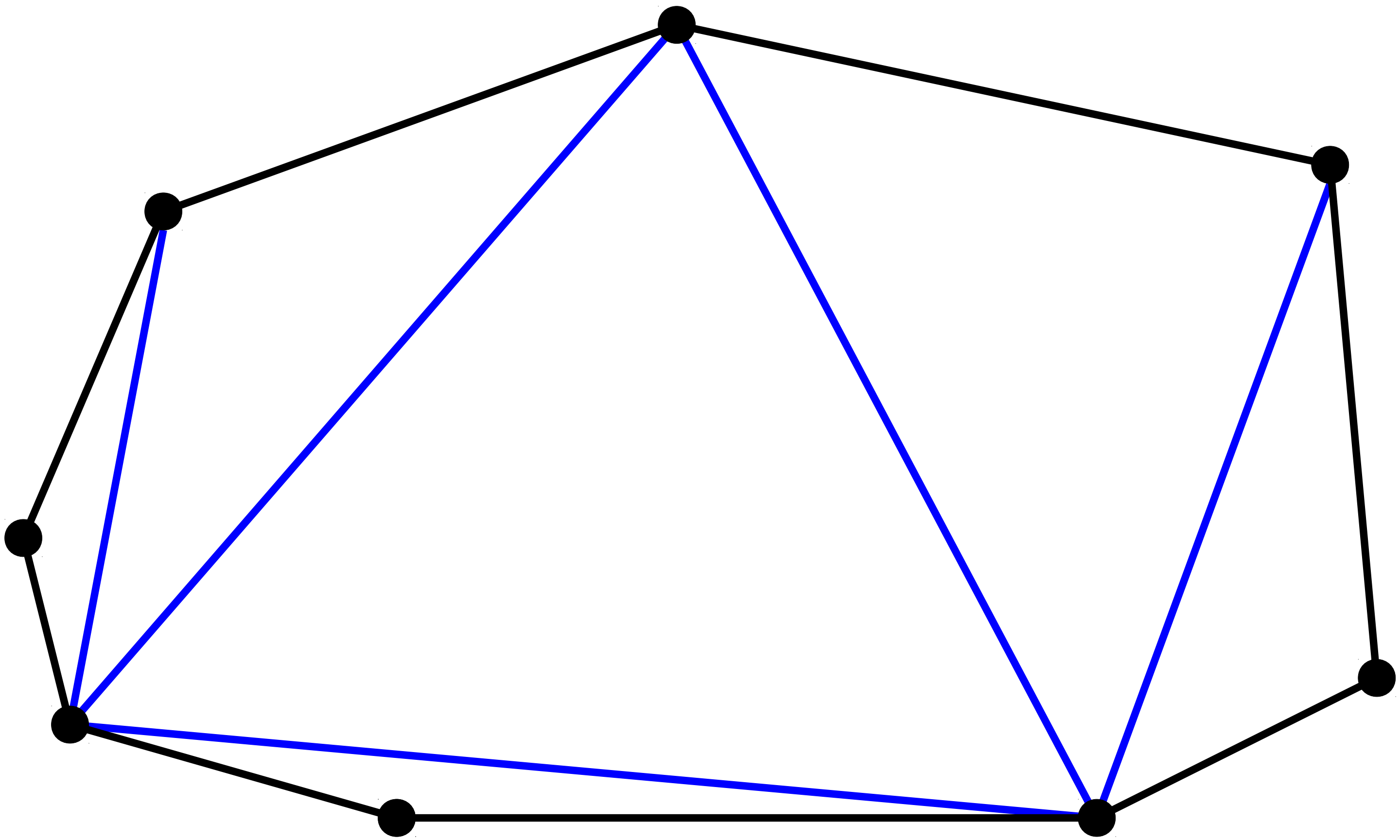
Complete Induction

- If the following are true:
  - $P(0)$ is true, and
  - If $P(0)$, $P(1)$, $P(2)$, …, $P(n)$ are true, then $P(n+1)$ is true as well.
- Then $P(n)$ is true for all $n \in \mathbb{N}$.
- This is called the **principle of complete induction** or the **principle of strong induction**.
  - (A note: this also works starting from a number other than 0; just modify what you're assuming appropriately.)

Proof by Complete Induction

- State that your proof works by complete induction.
- State your choice of $P(n)$.
- Prove the base case: state what $P(0)$ is, then prove it using any technique you'd like.
- Prove the inductive step:
  - State that for some arbitrary $n \in \mathbb{N}$ that you're assuming $P(0), P(1), \ldots, P(n)$ (that is, $P(n')$ for all natural numbers $0 \leq n' \leq n$.)
  - State that you are trying to prove $P(n + 1)$ and what $P(n + 1)$ means.
  - Prove $P(n + 1)$ using any technique you'd like.

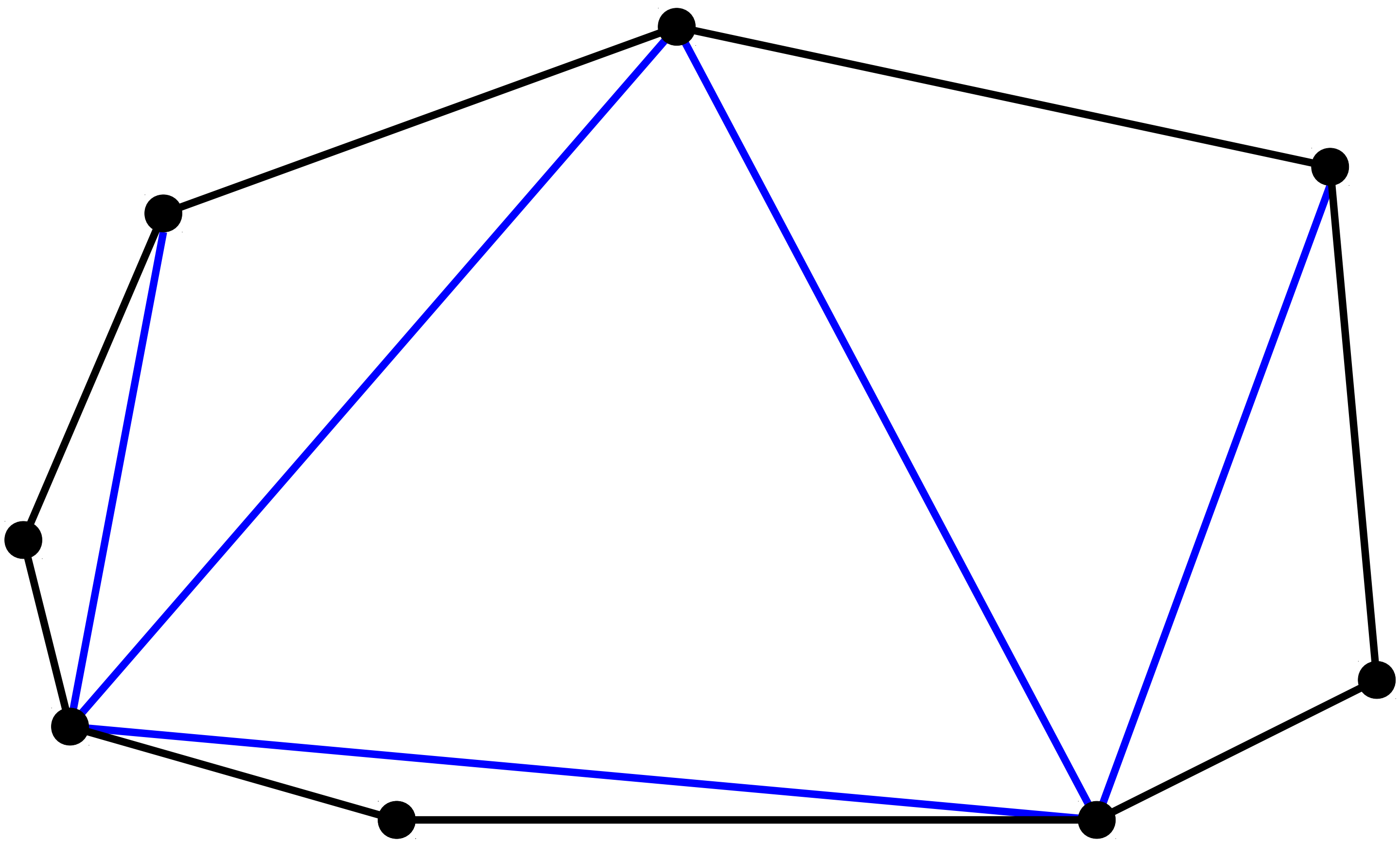
Example: Polygon Triangulation



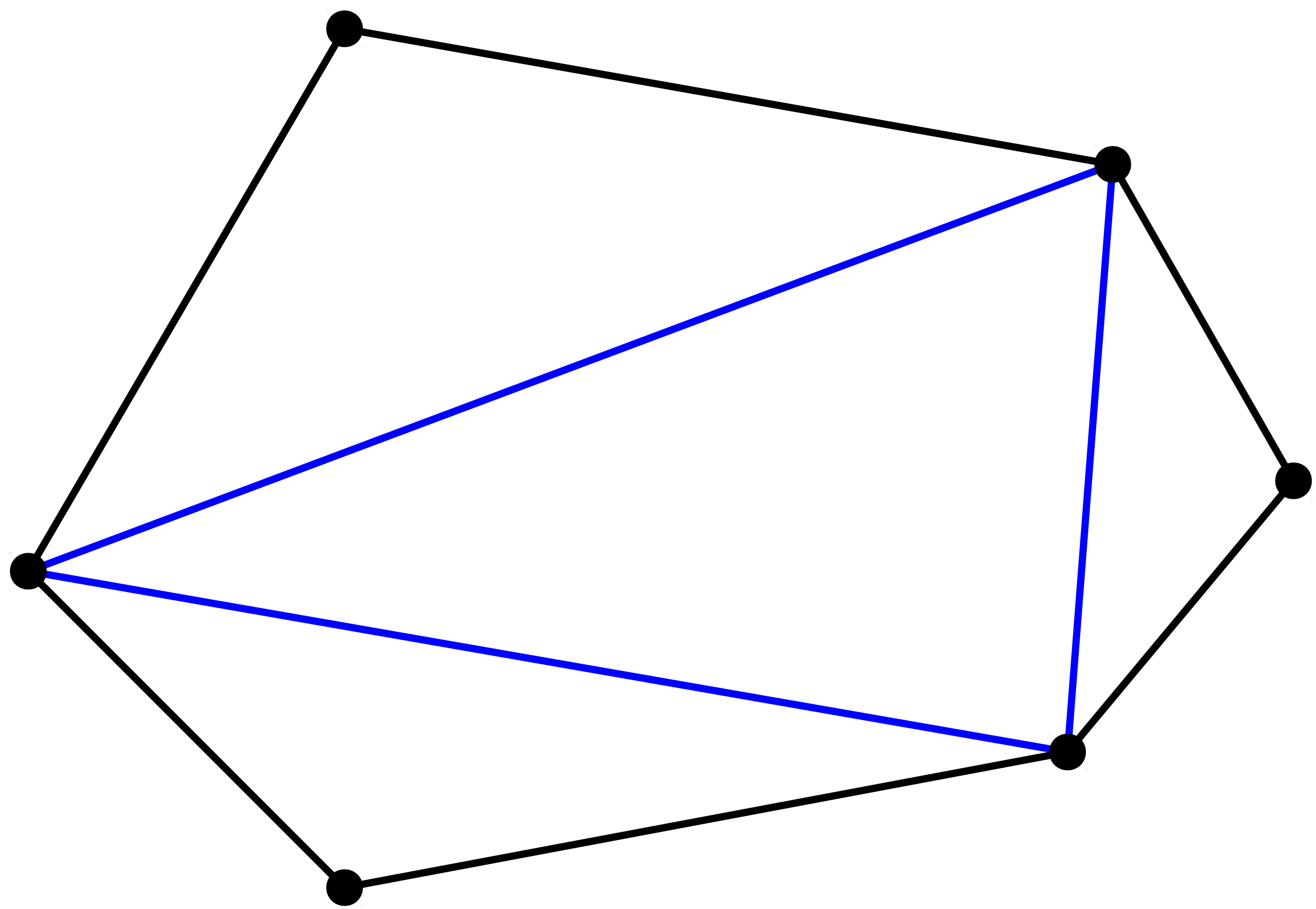
Polygon Triangulation

- Given a convex polygon, an **elementary triangulation** of that polygon is a way of connecting the vertices with lines such that
  - No two lines intersect, and
  - The polygon is converted into a set of triangles.
- Question: How many lines do you have to draw to elementarily triangulate a convex polygon?

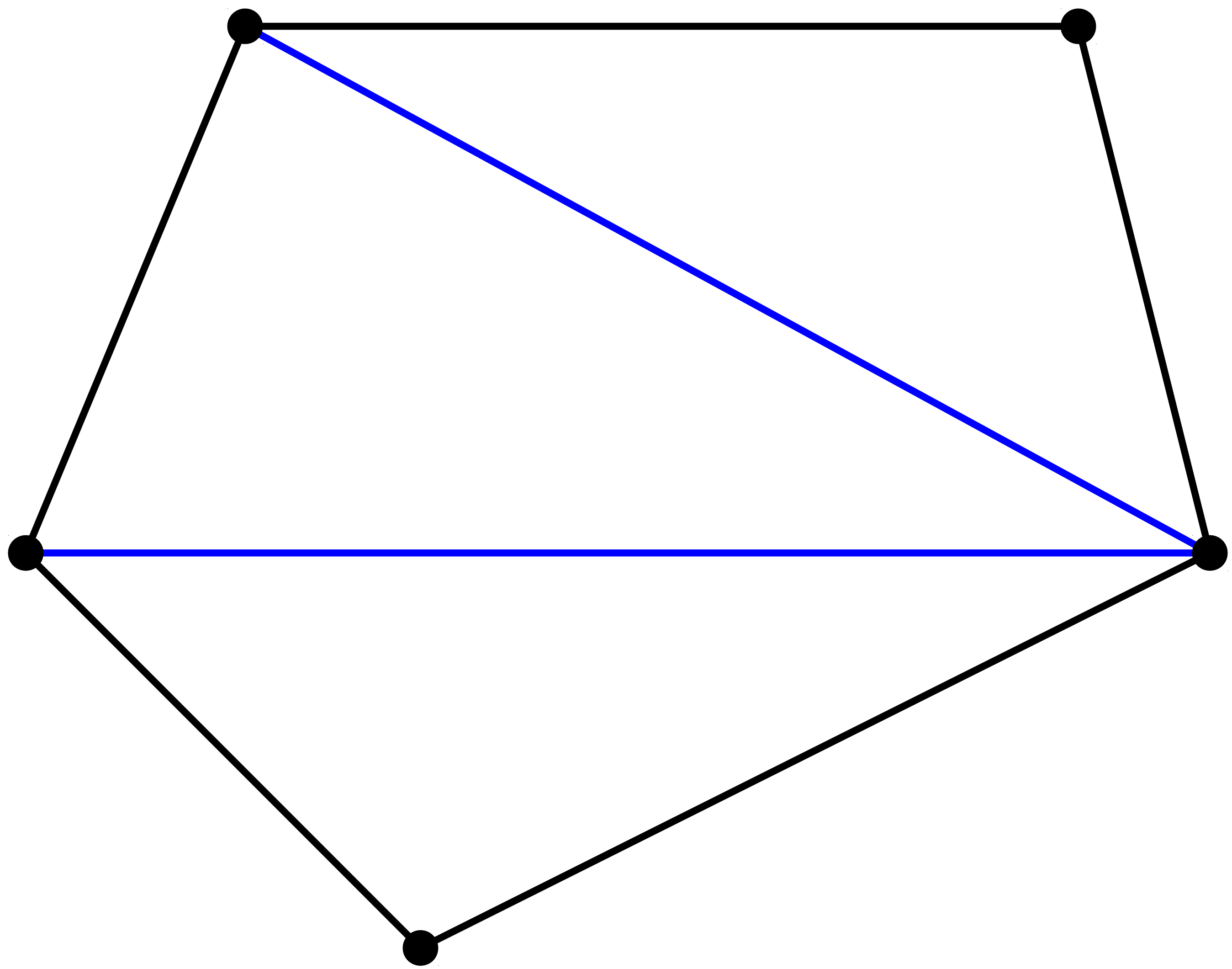
Elementary Triangulations



Elementary Triangulations



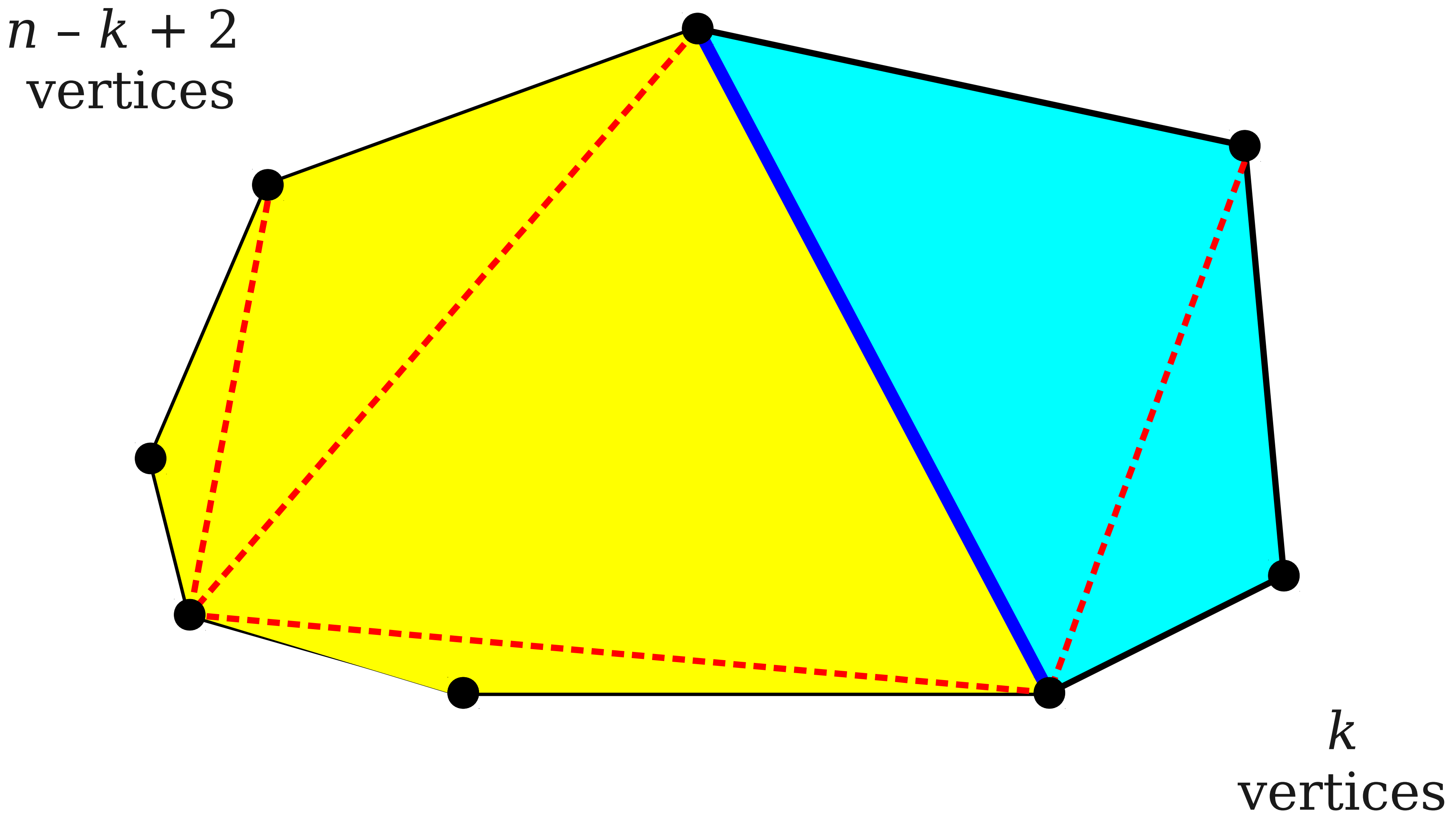
Elementary Triangulations



Some Observations

- Every elementary triangulation of the same convex polygon seems to require the same number of lines.
- The number of lines depends on the number of vertices:
  - 5 vertices: 2 lines
  - 6 vertices: 3 lines
  - 8 vertices: 5 lines
- **Conjecture:** Every elementary triangulation of an n-vertex convex polygon requires $n - 3$ lines.

Elementary Triangulations



Theorem: Every elementary triangulation of a convex polygon with n vertices requires $n - 3$ lines.

Proof: By complete induction. Let $P(n)$ be "every elementary triangulation of a convex polygon requires n–3 lines." We prove $P(n)$ holds for all $n \geq 3$. As a base case, we prove $P(3)$: elementarily triangulating a convex polygon with three vertices requires no lines. Any polygons with three vertices is a triangle, so any elementary triangulation of it will have no lines.

For the inductive step, assume for some $n \geq 3$ that $P(3)$, $P(4)$, …, $P(n)$ are true. This means any elementary triangulation of an n'-vertex convex polygon, where $3 \leq n' \leq n$, uses n'–3 lines. We prove $P(n+1)$: any elementary triangulation of any $(n+1)$-vertex convex polygon uses n–2 lines.

Let A be an arbitrary convex polygon with n+1 vertices. Pick any elementary triangulation of A and select an arbitrary line in that triangulation. This line splits A into two smaller convex polygons B and C, which are also triangulated. Let k be the number of vertices in B, meaning C has $(n+1)–k+2 = n–k+3$ vertices. By our inductive hypothesis, any triangulations of B and C must use k–3 and n–k lines, respectively. Therefore, the total number of lines in the triangulation of A is $n–k+k–3+1 = n–2$. Thus $P(n+1)$ holds, completing the induction. ■

Using Complete Induction

- When is it appropriate to use complete induction in contrast to standard induction?
- Depends on the proof approach:
  - Typically, standard induction is used when a problem of size $n + 1$ is reduced to a simpler problem of size n.
  - Typically, complete induction is used when the problem of size $n + 1$ is split into multiple subproblems of unknown but smaller sizes.
- It is never “wrong” to use complete induction. It just might be unnecessary. We suggest writing drafts of your proofs just in case.

Summary

- Induction doesn't have to start at 0. It's perfectly fine to start induction later on.
- Induction doesn't have to take steps of size 1. It's not uncommon to see other step sizes.
- Induction doesn't have to have a single base case.
- Complete induction lets you assume all prior results, not just the last result.

Next Time

- **Graphs**
  - Representing relationships between objects.
  - Connectivity in graphs.
  - Planar graphs.