Invariance

Invariants are simple properties that do not change as a process continues. Picking the right invariant can give you insight about how the process progresses and may give you criteria for the final state. Here's an example problem:

Question: Alice writes the numbers 1, 2, 3, 4, 5, and 6 on a blackboard. Bob selects two of these numbers, erases both of them, and writes down their sum on the blackboard. For example, if Bob chose the numbers 3 and 4, the blackboard would contain the numbers 1, 2, 5, 6, and 7. Bob continues until there is only one number left on the board. What are the possible values of that number? [1]

Solution: Consider the sum of all the numbers on the board. This is invariant under the operation that Bob does: namely removing 2 numbers and replacing them with their sum. For instance, if Bob chooses to erase the numbers a and b, he will write $a+b$ on the blackboard, making the new total sum $n-a-b+(a+b)=n$, so this is indeed an invariant. This means that at any time during the process, the sum of the numbers on the blackboard will be $n=1+2+3+4+5+6=21$, which means that when there's only one number remaining, the final number must be 21.

Monovariance

Monovariants are properties that may change as a process continues but the change is in a predictable direction. For instance, the monovariant may always increase, or always decrease, or is always odd.

Question: (Week 4 Putnam Class) n children are playing in groups on a playground (a group might have only one child in it). Every minute, one child leaves their current group and joins a group that has at least as many children as their previous group. Prove that eventually all of the children are playing in one huge group.

Solution: Let s_i denote the size of the group that the i^{th} child is in. Let S denote the sum of s_i for every child from 1 to n. When one child moves from a group of size s_i to a group of size s_j (where $s_i \geq s_j$), then S changes the following way:

- Before the movement, there are s_i kids in the source group and s_j kids in the target group. So these 2 groups contribute $s_i \times s_i + s_j \times s_j = s_i^2 + s_j^2$ to S
- After the movement, there are $s_i - 1$ kids in the source group and $s_j + 1$ kids in the target group. So these 2 groups contribute $(s_i - 1) \times (s_i - 1) + (s_j + 1) \times (s_j + 1) = s_i^2 + s_j^2 + 2s_j - 2s_i + 2$ to S.

However, $s_j \geq s_i$, so the new S is atleast 2 more than the old S. Thus S is a monovariant that increases every time a child moves. The only way S stops increasing is if there is no way a child could move, which means there is only one group left.

[1] Source https://brilliant.org/wiki/invariant-principle-definition/