Jacobians of Regular Matroids

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Matroid

Definition

A *Matroid* $M = (E, I)$ is a finite set E, along with I a collection of "independent" subsets of E, such that

- $\emptyset \in I$
- If $A \in I$, $B \subseteq A$, then $B \in I$
- $\forall A, B \in I$, $|A| > |B|$, then $\exists x \in A \setminus B$ such that $\{x\} \cup B \in I$

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Example

Let G be a connected graph.
$E = \{e_1, e_2, ..., e_m\}$ is the set of edges
Elements in I are all subsets of E that don't contain a cycle
Note that spanning trees are maximally independent subsets.

Regular Matroid

Definition

A matroid M is *realizable* over a field K if M can be represented by columns of a matrix (treated as vectors) over K.

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Fact

A matroid M is regular $\Leftrightarrow$ M can be represented by a weakly unimodular matrix A.

Weakly Unimodular

Definition

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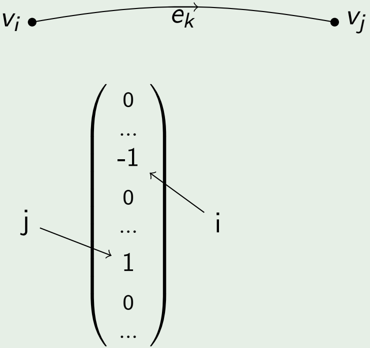
Example

Any graphic matroid is regular.



Jacobian of Regular Matroid

Let M be a regular Matroid, and A be a weakly unimodular matrix representing M.

Definition

Cycle lattice $L = Ker(A) \cap \mathbb{Z}^m$ where m = number of columns

Note: If M is graphic, L = lattice of integral flows $= H_1(G, \mathbb{Z})$

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Definition

$L^{\#} = \{x \in Ker(A) \colon \langle x, v \rangle \in \mathbb{Z}, \forall v \in L\}$

One can easily see $L \subseteq L^{\#}$

Fact

$\pi(e_i) \in L^{\#}$ *because* $\langle \pi(e_i), C \rangle = \langle e_i, \pi(C) \rangle = \langle e_i, C \rangle \in \mathbb{Z}$ *where* $C \in L$

Jacobian of Regular Matroid

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Definition

The *Jacobian* is defined as $Jac(M) = L^{\#}/L$

where $L = Ker(A) \cap \mathbb{Z}^m$ where $m =$ number of columns
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Fact

$Jac(M)$ is independent of the choice of A.

In graphs, this is the $Jac(G)$ or $Pic^0(G)$

Bijection between Jacobian and Bases

Theorem

$|Jac(M)| = Number\ of\ Bases$
where bases are maximally independent sets

- This is *essentially* a generalization of Kirchhoff's Matrix-Tree theorem from graphs to regular matroids, and can be proved similarly.
- Is there an explicit bijection?
- Is there a bijective proof?

Bijection between Jacobian and Bases

(REU 2014) Explicit bijection between *Bases* and $Jac(M)$

Input: Regular matroid M, any weakly unimodular matrix representation, A. Basis T

Algorithm:

- If $e \in T$, we 'keep' e iff e disagrees with the largest edge in $cut(T, e)$
- If $e \notin T$, we 'keep' e iff e agrees with the largest edge in $cycle(T, e)$

where the 'largest edge' is determined by its column number in A.

Output: $\sum_{e:\ kept} \pi(e) + L \in Jac(M)$

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$Cycle(T, e)$ is the fundamental cycle (minimally dependent set) formed by some elements in T and the element e.
$Cut(T, e)$ is the minimal set formed by some elements in $E \setminus T$ and the element e, such that it has non-empty intersection with every basis of M.

This is a new algorithm, *even for graphs*.

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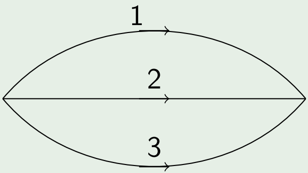
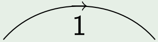
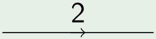


Spanning Trees:





Output: $\pi(e_2) + \pi(e_3) + L$ $\quad$ $\pi(e_3) + L$ $\quad$ L

Geometric Cycle Orientations

Theorem

The algorithm with

- *Keep* $e \in T$ *if* e *disagrees with the largest edge in* $Cut(T, e)$
- *Keep* $e \notin T$ *if* e *agrees with* $Cycle(T, e)$ *given by a "geometric orientation" of cycles.*

is a bijection.

Geometric Cycle Orientations

Made computationally precise as follows:
Assign weights $\alpha_i \in \mathbb{R}$ to each edge e_i. For a cycle C, pick a particular direction of traversal.
C is oriented in this direction if $\sum_{i=1}^{m} \alpha_i \langle e_i, C \rangle$ is positive.
Otherwise, C is directed in the opposite direction.

Fact

This assignment of weights to orient cycles generates all possible geometric cycle orientations

Example

Orienting cycles along the largest edge is geometric.
Choosing weights $\alpha_i = 2^i$ generates this orientation.

Project 1

To compute an inverse map for the algorithm, which has only a geometric proof.

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Recursive Approach:
Say we can determine if edge e is in the tree or not.
If $e \in T$ run the inverse on M / e
If $e \notin T$ run the inverse on $M \setminus e$
This deletion or contraction can change the orientation of cuts and cycles. We have constructed an rule to update the edge weights after deletion or contraction to maintain the orientations of cycles and cuts.

However, further ideas are needed as the inverse is subtle.

Project 2

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We define *geometric cut orientations* similar to cycles by choosing m real numbers β_i and orienting cuts such that $\sum_{i=1}^{m} \beta_i \langle e_i, C \rangle > 0$

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We define *geometric cut orientations* similar to cycles by choosing m real numbers β_i and orienting cuts such that $\sum_{i=1}^{m} \beta_i \langle e_i, C \rangle > 0$

Fix a set of m generic real numbers β_i for cuts and m generic real numbers α_i for cycles

Conjecture

The algorithm with

- Keep $e \in T$ if e agrees with Cut(T, e) oriented by β_i 's
- Keep $e \notin T$ if e agrees with Cycle(T, e) oriented by α_i 's

is a bijection.

Partial results for certain classes of β_i
However, this algorithm is NOT geometric in the cycle space, so new ideas are needed. We conjecture that it is geometric in a higher space.

Research in progress

- Proving a generalized bijection.
- Showing the bijection is geometric in the space of edges
- Computing an inverse map.
- And many more interesting directions...

Acknowledgements

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