Homotopy Groups of Embedding Spaces

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Embedding spaces

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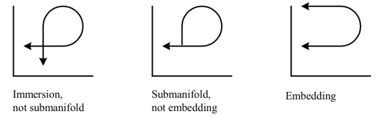

Source: "Mathematics for Physics, An Illustrated Handbook"

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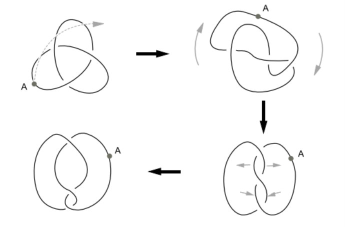

Source: Skopenkov https://arxiv.org/pdf/2001.01472v1.pdf

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Let M denote a compact 4-manifold with boundary.
The main embedding space in this talk will be:

$\mathrm{Emb}_\partial(I, M) := \{$ embeddings $f : I \to M |$ end points of I are fixed in $M\}$

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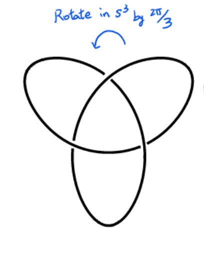

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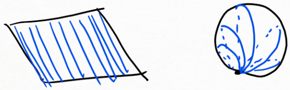

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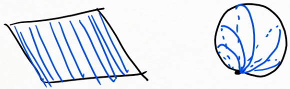
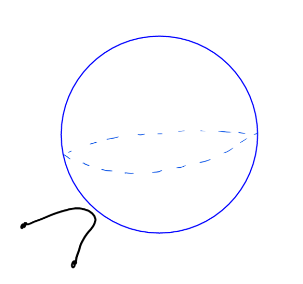


Budney and Gabai (2021) construct 'knotted balls' in $S^1 \times B^3$ by viewing them as elements of $\pi_2(\mathrm{Emb}_\partial(I, S^1 \times B^3))$.

Lasso Operations in 4-manifolds

Embedded intervals lasso around an embedded S^2 in M:



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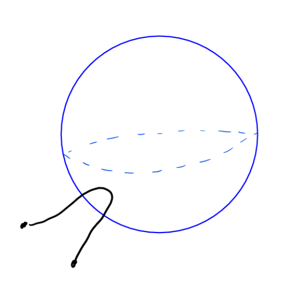

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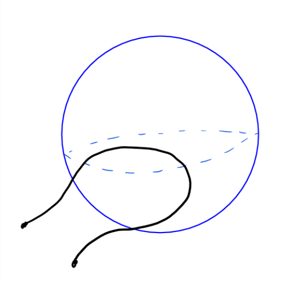

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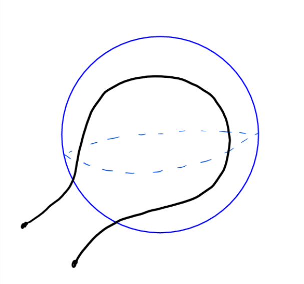

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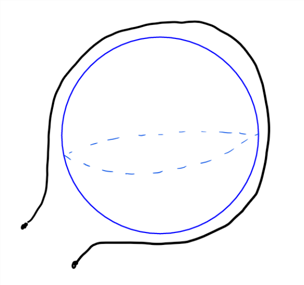

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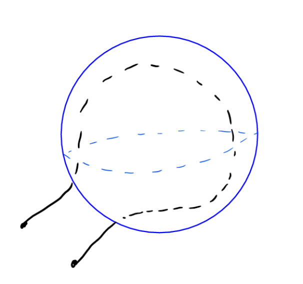

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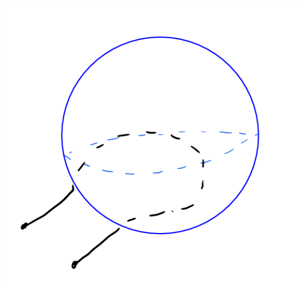

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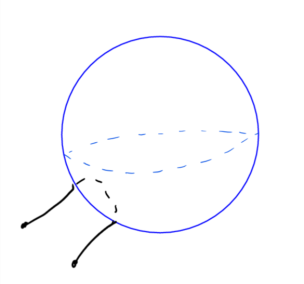

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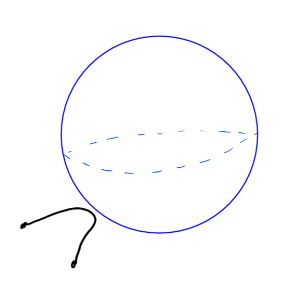

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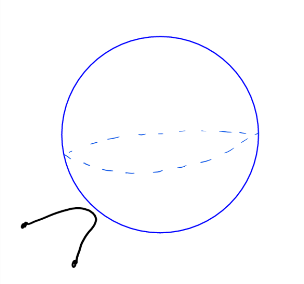

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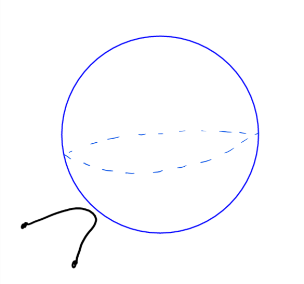

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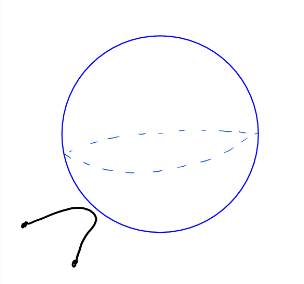

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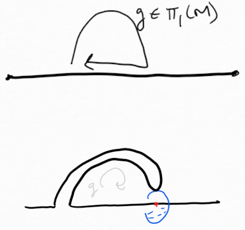

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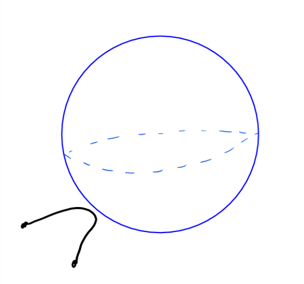

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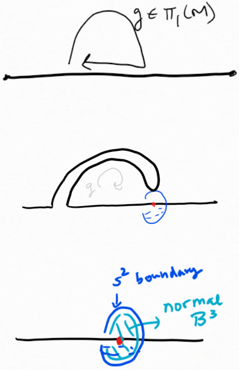

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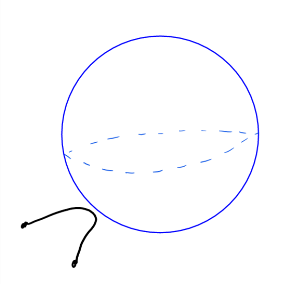

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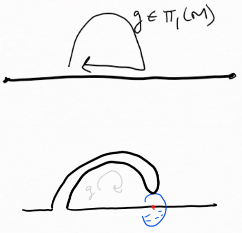

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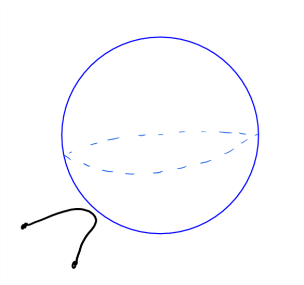

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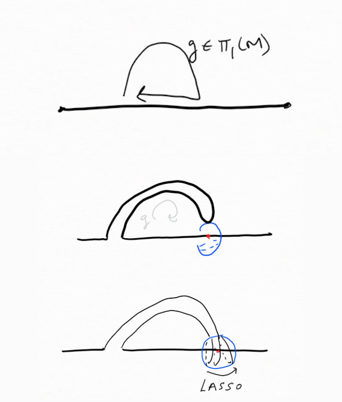

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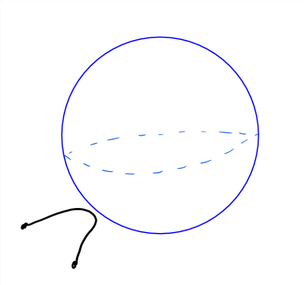

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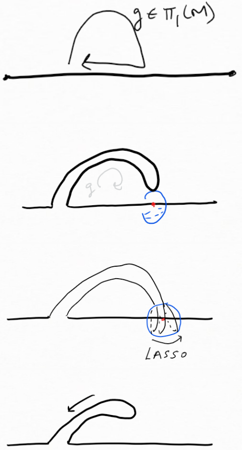

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$$\begin{tikzcd} S^1 \times S^1/S^1 \vee S^1 \arrow[rr] \arrow[dr] & & \mathrm{Emb}_\partial(I, M) \\ & S^2 \arrow[ur] & \end{tikzcd}$$

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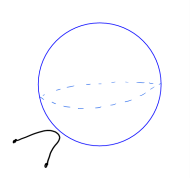

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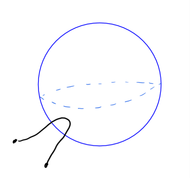

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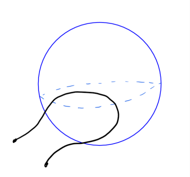

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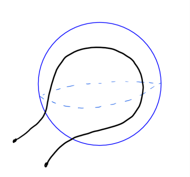

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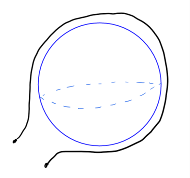

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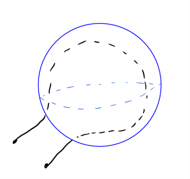

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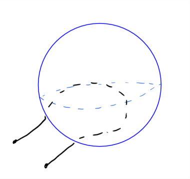

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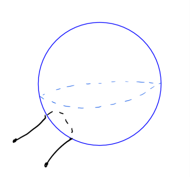

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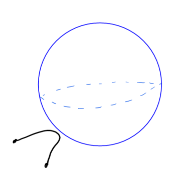

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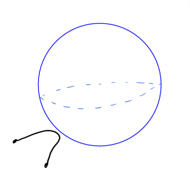

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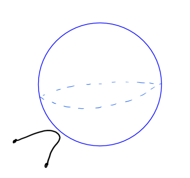

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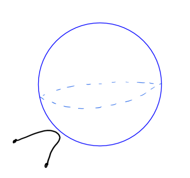

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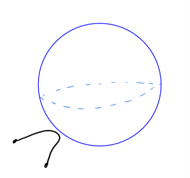

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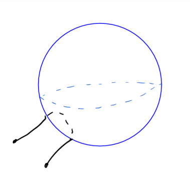

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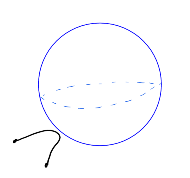

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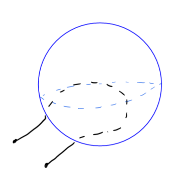

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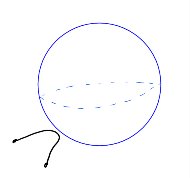

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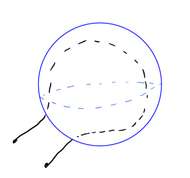

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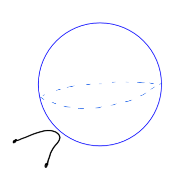

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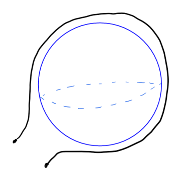

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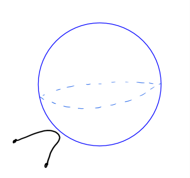

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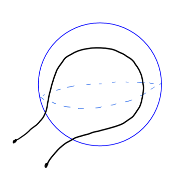

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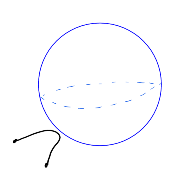

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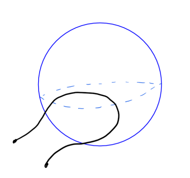

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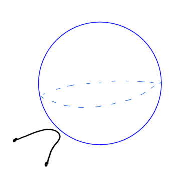

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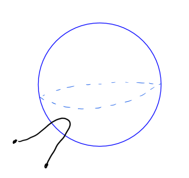

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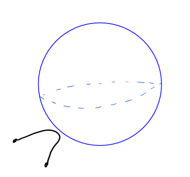

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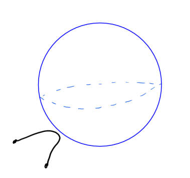

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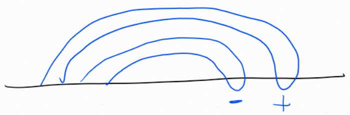

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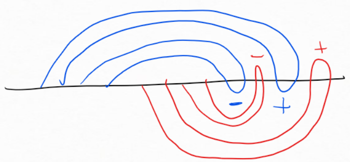

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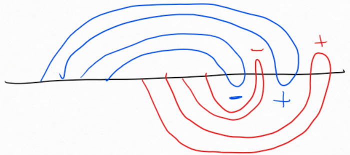

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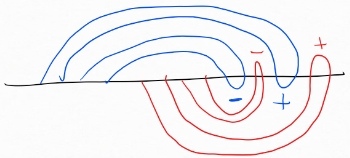

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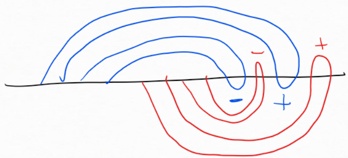

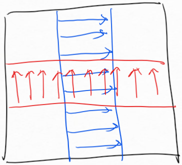

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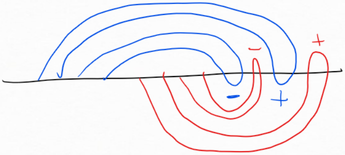

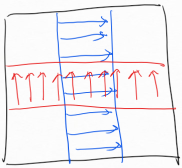

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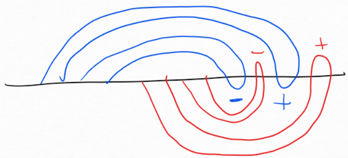

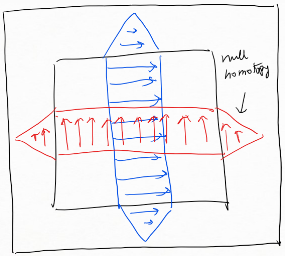

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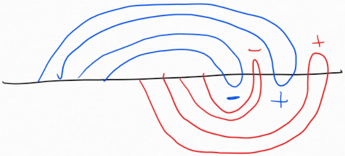

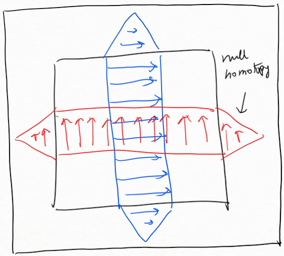

We have thus constructed a map $S^2 \to \mathrm{Emb}_\partial(I, M)$

Configuration spaces

Configuration spaces

Definition

The *n*-point configuration space of a manifold M is

$$C_n(M) := \{(p_1, \cdots p_n) \in M^n | p_i \neq p_j \text{ for } i \neq j\}$$

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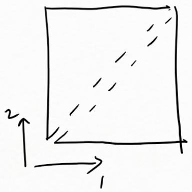

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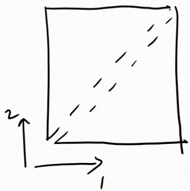

$C_n(I)$ is homoeomorphic to the (disjoint union of) n-simplex Δ^n

From embedding spaces to configuration spaces

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\begin{tikzcd}
\pi_k(Emb_\partial(I, M)) \arrow[r] \arrow[ddr] & \pi_k(Map^{sp}(C_n(I), C_n(M))) \arrow[d] \\
& Map^{sp}(I^k \times I^n, C_n(M))/\sim \arrow[d, "\text{collapse boundary}"] \\
& \pi_{n+k}(C_n(M))/\langle R \rangle
\end{tikzcd}
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Theorem (S.S.)

There exist nontrivial elements of $\pi_3(\mathrm{Emb}_\partial(I, M))$, arising from a nontrivial map to $\pi_7(C_4(M))/\langle R\rangle$.

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Thank You!