Cusps of Hyperbolic Knots

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Hyperbolic Knots

Definition

A *hyperbolic knot or link* is a knot or link whose complement in S^3 is a 3-manifold that admits a complete hyperbolic structure.

This gives us a very useful invariant for hyperbolic knots: Volume (V) of the hyperbolic knot complement.

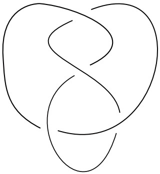
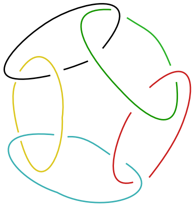


Figure 8 Knot
Volume=2.0298...



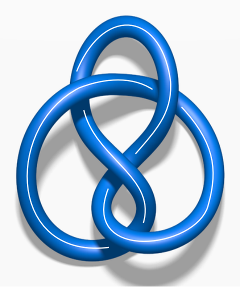
5 Chain
Volume=10.149.....

Cusp

Definition

A *Cusp* of a knot K in S^3 is defined as an open neighborhood of the knot intersected with the knot complement: $S^3 \setminus K$

Thus, a cusp is a subset of the knot complement.



Cusp Volume

The *maximal cusp* is the cusp of a knot expanded as much as possible until the cusp intersects itself.

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Definition

The *Cusp Volume* of a **hyperbolic** knot is the volume of the maximal cusp in the hyperbolic knot complement.

Since the cusp is a subset of the knot complement, the cusp volume of a knot is always less than or equal the volume of the knot complement.

Fact

The cusp volume of the figure-8 knot is $\sqrt{3}$

Cusp Volume of Links

How should we expand cusps in a link to achieve maximal volume?

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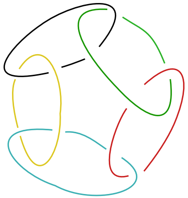
Fact

The optimal cusp volume of links will be achieved by first, maximizing a particular cusp till it touches itself, then a particular second cusp till it touches itself or the first cusp and so on...

Fact

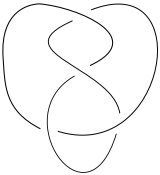
The optimal cusp volume of the minimally twisted 5-chain is $5\sqrt{3}$ *where the individual cusps have volumes:* $4\sqrt{3}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}$ *respectively.*



Cusp Density

Definition

Cusp Density (D_c) of a knot or link is the ratio: $\frac{CV}{V}$ where CV is the total cusp volume and V is the hyperbolic volume of the complement.



Volume=2.0298...
Cusp Volume=$\sqrt{3}$
Cusp Density=0.853

Volume=10.149...
Cusp Volume $= 5\sqrt{3}$
Cusp Density=0.853

Main Theorems

Fact

(Böröczky 1978) The highest cusp density a hyperbolic manifold can have is 0.853..., *the cusp density of the figure-8 knot and the minimally twisted 5-chain.*

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Theorem

(SMALL 2016) For any $x \in [0, 0.853...]$, *there exist hyperbolic* ***link*** *complements with cusp density arbitrarily close to* x.

Theorem

(SMALL 2016) For any $x \in [0, 0.6826...]$, *there exist hyperbolic* ***knot*** *complements with cusp density arbitrarily close to* x.

Constructions

Three constructions provide essential elements to the proofs:

- Dehn Filling
- Cyclic covers
- Belted sums

Dehn Filling

Definition

$(1, q)$ Dehn filling on an **unknotted** component of a link complement gives a link complement of the original link, without the trivial component, and the strands through it twisted q times.

(1,2) Dehn Filing

Dehn Filling

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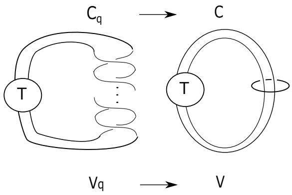
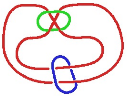
Fact

As $q \to \infty$, the Volume and Cusp volume approaches that of the original link before Dehn filling

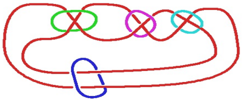


n-fold Covers

The following construction is taking "n-fold cyclic covers" of a tangle across a twice punctured disc.



Twisted Borromean Rings



3-fold cover

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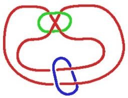

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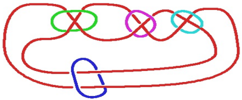


3-fold cover

We know that volume of an $n-$fold cover L_n is n times the volume of the original link L.

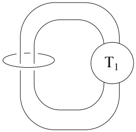
$$V(L_n) = nV(L)$$

In fact, the same holds for cusp volumes and we can see this as a multiplication of the fundamental domain.

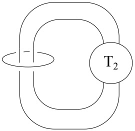
$$CV(L_n) = nCV(L)$$

Belted Sums

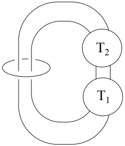
The following construction is taking a belted sum of links cross a twice punctured disc



L_1



L_2



Belted Sum: L_{1+2}

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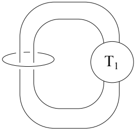

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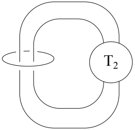
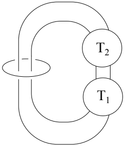


L_2



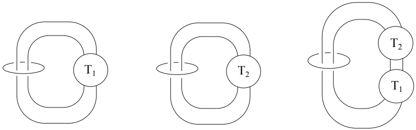
Belted Sum: L_{1+2}

$$V(L_1) + V(L_2) = V(L_{1+2})$$

We know that volume of a belted sum is the sum of the volumes of the original 2 links. What happens to cusp volumes?

Belted Sums

Cusp volume doesn't add so nicely.



L_1 L_2 L_{1+2}

Theorem* (SMALL 16) $*$Under the absence of *poking*

$$RCV(L_{1+2}) = RCV(L_1) + \left(\frac{m_1}{m_2}\right)^2 RCV(L_2)$$

m_1 and m_2 are *meridian lengths of the maximal cusps of* L_1 *and* L_2

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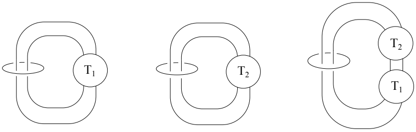

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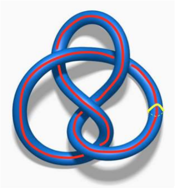
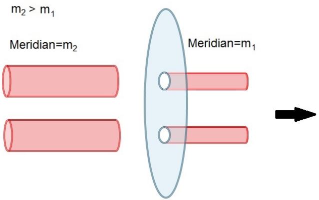
The boundary of a cusp is a torus.

The meridian is marked in yellow and the longitude in red.



Belted Sums

Heuristic Proof:



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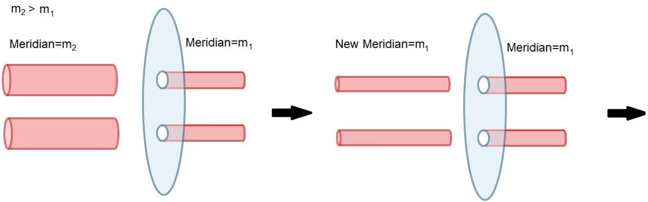

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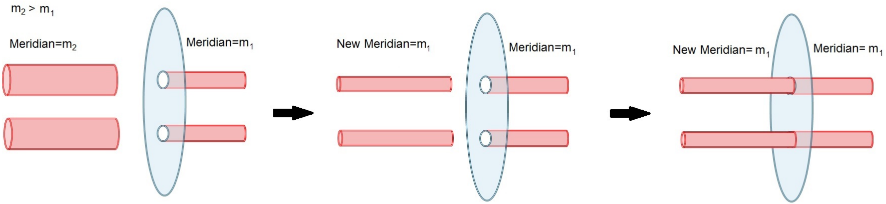

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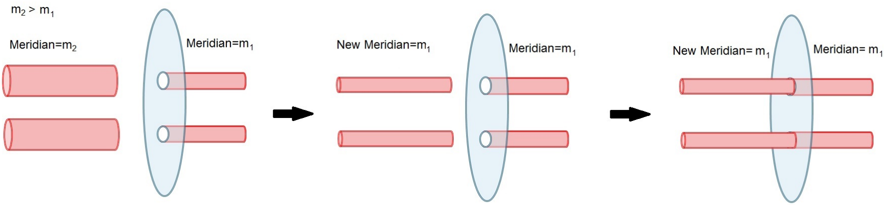

$$\text{Volume of Cusp} \propto \text{meridian}^2$$

$$RCV(L_2^{new}) = \left(\frac{m_1}{m_2}\right)^2 RCV(L_2)$$

$$RCV(L_{1+2}) = RCV(L_1) + \left(\frac{m_1}{m_2}\right)^2 RCV(L_2)$$

Finishing the proof

We find two tangles augmented by a link with volumes V_1 and V_2, restricted cusp volume RCV_1 and RCV_2 and meridians m_1, m_2.

We take $'a'$ fold covers of one and *belt sum* with a $'b'$ fold cover of the other to get a link.

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We take $'a'$ fold covers of one and *belt sum* with a $'b'$ fold cover of the other to get a link.

$$RCV = a\mathrm{RCV}_1 + b\left(\frac{m_1}{m_2}\right)^2 \mathrm{RCV}_2 \quad \text{and} \quad V = a\mathrm{V}_1 + b\mathrm{V}_2$$

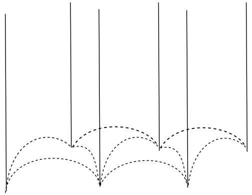
Finally, perform high *Dehn filling* to the unknotted component to get a knot with cusp density close to $\dfrac{\left(\frac{a}{b}\right)RCV_1 + \left(\frac{m_1}{m_2}\right)^2 \mathrm{RCV}_2}{\left(\frac{a}{b}\right)V_1 + V_2}$.

Choosing the right links gives us cusp density dense in $[0, 0.6826..]$

Covering space $\mathbb{H}^3$

Fact

Complements of hyperbolic knots lift to several copies of polyhedra in $\mathbb{H}^3$



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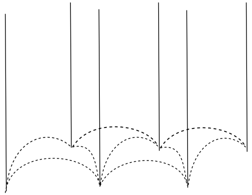
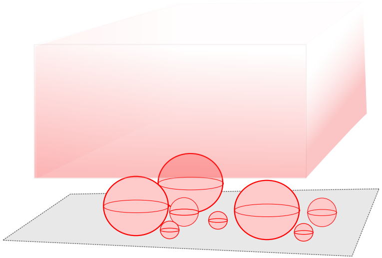


Fact

Cusps of hyperbolic knots lift to disjoint unions of horoballs in the upper half model of hyperbolic space



Cusp Diagrams

Where do different parts of the cusp go in this horoball diagram?

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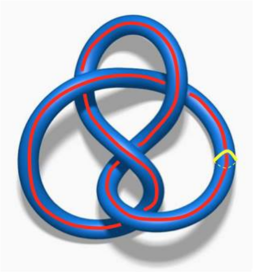
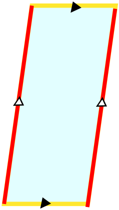
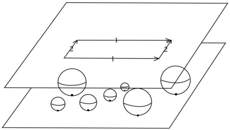
Where do different parts of the cusp go in this horoball diagram?

We first look at the boundary of the cusp. When we cut along the **meridian** and along the **longitude**, we get a rectangle with the identifications as shown. This corresponds to a fundamental domain on the boundary of the horoball centered at ∞.

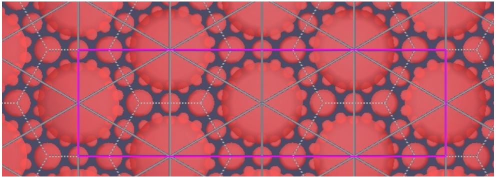






Cusp Diagrams-Example of figure 8

View from ball at infinity



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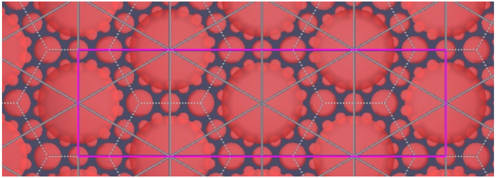
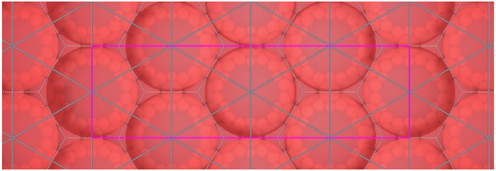
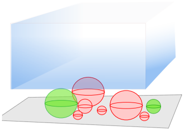


When the cusp is maximized

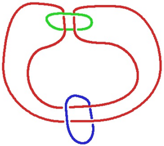


Horoball Diagrams of Links

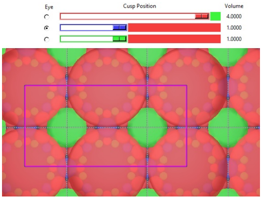
Consider the Borromean Rings. The 3 cusps lift to horoballs in $\mathbb{H}^3$. Each horoball (tangent to $z = 0$ or centered at ∞) is actually centered at the same point at infinity. This gives us a **choice** to view the horoball diagram from a particular horoball centered at infinity in the diagram (the one that looks like space enclosed above a plane).



Horoball Diagram



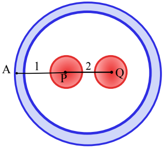
The Borromean Rings



Cusp Diagram looking from the blue cusp at infinity

Links and Twice Punctured Discs

Consider a trivial component around 2 strands. This bounds a twice punctured disc in the complement as shown.



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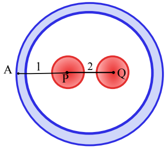
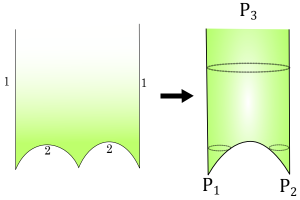


A theorem by Adams states that a twice punctured disc in a hyperbolic 3-manifold always lifts to a totally geodesic surface in the upper half space, as shown.



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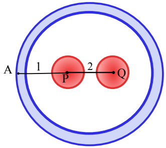

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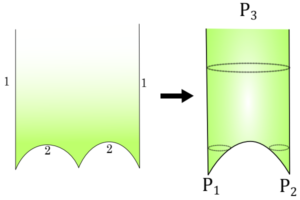

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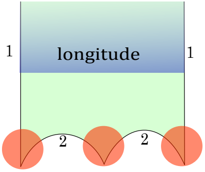
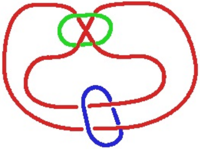
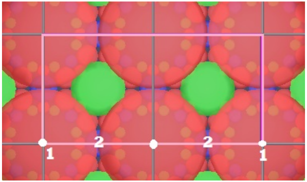


Horoball Diagram

Twice Punctured Discs in horoball Diagrams



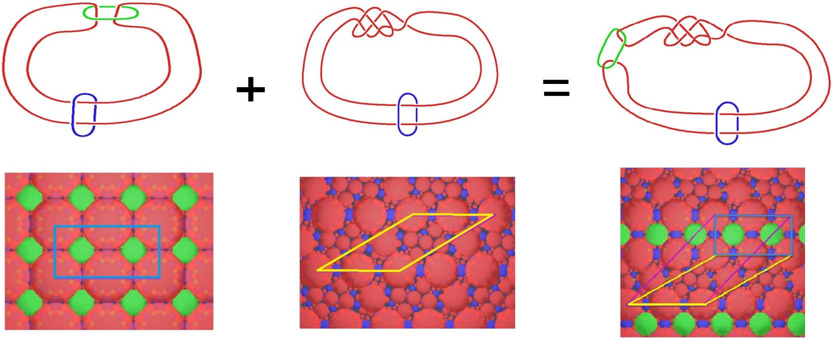
Twisted Borromean Rings



Cusp Diagram of the twice punctured looking from the blue cusp

The twice punctured disc, when viewed from the blue cusp at infinity surfaces can be seen as passing through the longitude of the blue cusp in the horoball diagram.

Horoball diagram of Belted Sums



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References

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