Modular Forms and Jacobi's Four Square Theorem

Shruthi Sridhar

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1 Introduction

In 1770, Lagrange proved that any natural number n can be written as the sum of squares of 4 integers. In 1834, Jacobi proved the following remarkable result on the number of ways to write a number as a sum of 4 squares.

Theorem 1.1. *Let $n \in \mathbb{N}$. Define $r_k(n) = \#\{(a_1, a_2, ..., a_k) \in \mathbb{Z}^k \mid \sum a_i^2 = n\}$, the number of ways to write n as the sum of k squares. Then, $r_4(n) = \sum\limits_{d|n, d \notin 4\mathbb{Z}} 8d$*

This expository paper begins with a short survey of modular forms over $SL(2, \mathbb{Z})$ and its congruence subgroups and uses these ideas to prove Jacobi's four square theorem.

2 The action of $SL(2, \mathbb{Z})$ on $\mathbb{H}$

The upper half plane is defined as: $\mathbb{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$

Lemma 2.1. *The group $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ via* ***Fractional Linear Transformations (FLTs)*** *as follows. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$. Then, $\gamma(z) = \frac{az+b}{cz+d}$ for any $z \in \mathbb{H}$.*

Proof. We first check that FLTs take $\mathbb{H}$ to $\mathbb{H}$.

$$\begin{aligned} Im(\gamma(z)) &= Im\Big(\frac{az+b}{cz+d}\Big) \\ &= Im\Big(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\Big) \\ &= Im\Big(\frac{ac|z|^2 + (ad+bc)Re(z) + i(ad-bc)Im(z) + bd}{|cz+d|^2}\Big) \\ &= \frac{Im(z)}{|cz+d|^2} > 0 \end{aligned}$$

Let $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2, \mathbb{R})$. We have

$$\begin{aligned} \gamma_1(\gamma_2(z)) = \gamma_1\Big(\frac{a_2 z + b_2}{c_2 z + d_2}\Big) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2) z + c_1 b_2 + d_1 d_2} = (\gamma_1 \cdot \gamma_2)(z) \end{aligned}$$

$\square$

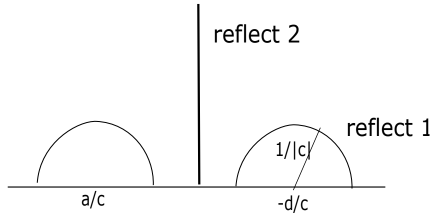

Figure 1: Isometric Circles

Because FLTs are a group action, we have that they are invertible. In fact, when $\mathbb{H}$ is treated as hyperbolic space with the metric $\mathrm{d}s = \frac{\sqrt{x^2+y^2}}{y}$, the group of isometries is precisely $PSL(2,\mathbb{R})$.

Now that we have defined an action by $SL(2,\mathbb{R})$, this induces an action by any of its subgroups. We will be especially interested in the action by the discrete subgroup $SL(2,\mathbb{Z})$.

A natural thing to look for is a fundamental domain for this action by $SL(2,\mathbb{Z})$. The translation matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ that maps $z \mapsto z+1$ is an element of the group. Thus the fundamental domain is inside the strip $|Re(z)| \leq \frac{1}{2}$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Let $z = \frac{-d}{c} + re^{i\theta}$. Then $\gamma(z) = \dfrac{a(\frac{-d}{c} + re^{i\theta}) + b}{c(\frac{-d}{c} + re^{i\theta}) + d} = \dfrac{\frac{-ad+bc}{c} + rae^{i\theta}}{cre^{i\theta}} = \frac{1}{rc^2}e^{-i\theta} + \frac{a}{c}$. Thus, γ reflects z first across the semicircle of radius $\frac{1}{|c|}$ centered at $\frac{-d}{c}$ and then reflects it about the line $x = \frac{a-d}{2c}$ as shown in Figure 1. We call the semi circles centered at $\frac{-d}{c}$ and $\frac{a}{c}$ as the 1st and 2nd isometric circles.

The region outside the 2nd isometric circle gets mapped inside the 1st isometric circle by γ. The region outside the 1st isometric circle gets mapped inside the 2nd isometric circle by γ^{-1}. So, the fundamental domain only needs one of these parts. We will choose to include the region outside the isometric circles. Thus the region (Figure 2) above all possible isometric circles and bounded in the strip will be a fundamental domain for the action by $SL(2,\mathbb{Z})$. Precisely, the fundamental domain is $\{z \in \mathbb{H} \big| |Re(z)| \leq \frac{1}{2} \text{ and } |z| \geq 1\}$

In fact, the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate $SL(2,\mathbb{Z})$ and the highest isometric circles come from the generators. This will be true for any subgroup of $SL(2,\mathbb{Z})$ acting on $\mathbb{H}$ as we will see later.

Before we move to defining modular forms, we will introduce the notion of cusps. When $SL(2,\mathbb{Z})$ acts on $\mathbb{H}$, it induces an action on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. (We will say γ takes ∞ to $\frac{a}{c}$). For any $p/q \in \mathbb{Q}$, the matrix $\begin{pmatrix} p & 0 \\ q & -p \end{pmatrix}$ $SL(2,\mathbb{Z})$ that takes $p/q \mapsto \infty$ and $\infty \mapsto p/q$. However, this

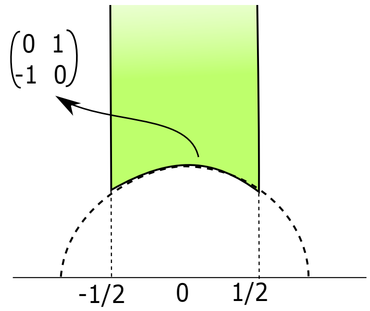

Figure 2: Fundamental domain for $SL(2,\mathbb{Z})$ acting on $\mathbb{H}$

may not be the case if we restrict the action to any subgroup Γ of $SL(2,\mathbb{Z})$. When we study actions by subgroups of $SL(2,\mathbb{Z})$, we will be interested in the orbits of $\mathbb{Q}\cup\infty$ when acted on by Γ. These orbits are called cusps.

3 Modular forms over $SL(2,\mathbb{Z})$

Modular forms are special holomorphic functions on $\mathbb{H}$ that satisfy some invariance under composition with FLTs.

Definition 3.1. Let k be any integer. A holomorphic function $f:\mathbb{H}\to\mathbb{C}$ is a **modular form** of weight k over $SL(2,\mathbb{Z})$ if

- $f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z)$ for all $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$.
- f is holomorphic at ∞.

We will make this notion of holomorphic at ∞ precise. We have $f(z+1)=(1)^k f(z)=f(z)$. Hence $f(z)=g(e^{2\pi iz})$ for some holomorphic $g:D\setminus\{0\}\to\mathbb{C}$. Thus $g(q)$ has a laurent series $g(q)=\sum\limits_{n=-\infty}^{\infty}a_nq^n$ where $q=e^{2\pi iz}$.

We say f is holomorphic at ∞ if $a_n=0$ for all $n<0$.

Example 3.2. Consider the functions $G_k(z)=\sum\limits_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{1}{(mz+n)^k}$ for $k\geq 3$. They will be modular forms of weight k.

$SL(2,\mathbb{Z})$ is generated by matrices $S=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, T=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, hence it is sufficient to check condition (1) for these matrices.

Clearly $G_k(z+1) = G_k(z)$. Also, $G_k\left(\frac{-1}{z}\right) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\frac{-1}{z}+n)^k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{z^k}{(nz+m)^k} = z^k G_k(z)$

It can be shown that $G_k(z)$ is holomorphic at infinity by showing it is bounded by the value at $\omega = e^{2\pi i/3}$. This is because, for z in the fundamental domain, $|mz+n|^2 = m^2 z\hat{z} + mnRe(z) + n^2 \geq m^2 + mn(-1/2) + n^2 = |m\omega + n|^2$. Moreover,

$$G_k(\infty) = \lim_{z\to\infty} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k} = \sum_{n\in\mathbb{Z}\setminus 0} \frac{1}{n^k} = 2\zeta(k)$$

We will see another way to define even weight modular forms as holomorphic differential forms on the space of orbits of the action. This is why the word $form$ in the name shows up.

Definition 3.3. Let k be any integer. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of weight $2k$ if

- $f(z)(dz)^k = f(\gamma(z))(d\gamma(z))^k$ for all $\gamma \in SL(2,\mathbb{Z})$. ($f(z)(dz)^k$ is seen as a k form on $\mathbb{H}$)
- f is holomorphic at ∞.

This tells us that we can see modular forms as holomorphic differential forms $f(z)(dz)^k$ on the space $\mathbb{H}/SL(2,\mathbb{Z})$. For instance, modular forms of weight 2 will be differential 1-forms $f(z)dz$ on the fundamental domain with appropriate identifications. This differential form is allowed to have a simple pole at ∞: Substituting $q = e^{2\pi iz}$, we get $f(z)dz = g(q)\frac{dq}{2\pi iq}$ which has a simple pole at $q = 0$.

This definition is compatible with the earlier definition as follows. We have $d\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2}dz$. Therefore, $f(gz) = (cz+d)^k f(z)$ if and only if $f(gz)d(gz)^k = (cz+d)^{2k} f(z)d(gz)^k = (cz+d)^{2k} f(z)(cz+d)^{-2k}(dz)^k = f(z)(dz)^k$. Similarly, this invariance precisely implies f transforms as in the definition of modular form by the same computation. So, $f(z)(dz)^k$ is $SL(2,\mathbb{Z})$ invariant if and only if $f(z)$ is a modular form of weight 2k.

4 Principle subgroups of $SL(2,\mathbb{Z})$

We will be interested in modular forms over certain special subgroups of $SL(2,\mathbb{Z})$ called congruence subgroups.

Definition 4.1. The principle congruence subgroup of level N is defined as:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

A congruence subgroup Γ of level N is any subgroup of $SL(2,\mathbb{Z})$ such that $\Gamma(N) \subset \Gamma \subset SL(2,\mathbb{Z})$.

Example 4.2. $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$

We have a more general definition of modular forms over congruence subgroups of $SL(2,\mathbb{Z})$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $f : \mathbb{H} \to \mathbb{C}$ holomorphic.

Define $f[\gamma]_k(z) = j(\gamma, z)^{-k} f(\gamma(z))$ where $j(\gamma, z) = cz + d$ and $\gamma(z) = \frac{az+b}{cz+d}$

Definition 4.3. Let k be any integer and Γ be a congruence subgroup. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if

- $f[\gamma]_k = f(z)$ for all $\gamma \in \Gamma$.
- $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL(2,\mathbb{Z})$.

The vector space of modular forms of weight k over Γ is denoted by $\mathcal{M}_k(\Gamma)$

It is not hard to show $j(\gamma\gamma', z) = j(\gamma, \gamma'(z))j(\gamma', z)$ and $f[\gamma\gamma']_k = (f[\gamma']_k)[\gamma]_k$.This makes checking the first condition for all $\gamma \in \Gamma$ equivalent to checking it for a generating set of Γ.
The 2nd condition checks holomorphic at infinity for all $\gamma \in SL(2,\mathbb{Z})$ to ensure that the function is well behaved at all cusps, not just ∞.

As in the case of modular forms over $SL(2,\mathbb{Z})$, we can view modular forms over congruence subgroups as differential k-forms on the space $\mathbb{H}/\Gamma$. When we look at weight 2 forms, the differential forms will be holomorphic on $\mathbb{H}/\Gamma$ with atmost simple poles at the cusps. The following section will find provide a function that is a modular form of weight 2 over the congruence subgroup $\Gamma_0(4)$

5 Theta Functions

We would like to have a generating function for the number of ways to write a number as a sum of k squares. One particular example of a theta function does the trick.

Define $\theta(z) = \sum_{m\in\mathbb{Z}} e^{2\pi i z m^2}$ on $\mathbb{H}$. This is a convergent series for $z \in \mathbb{H}$ because it is bounded by $2\sum_{n\geq 0} e^{-2\pi Im(z)n} = \frac{2}{1-e^{-2\pi Im(z)}}$. We notice that

$$\theta(z)^k = \sum_{n\in\mathbb{Z}} \left(\sum_{(a_1,\ldots a_k)|\sum a_i^2 = n} 1 \right) e^{2\pi i z n} = \sum_{n\in\mathbb{Z}} r_k(n) e^{2\pi i z n}$$

Thus $\theta(z)^4 = \sum\limits_{n\in\mathbb{Z}} r_4(n)e^{2\pi izn}$ and this is the required generating function.
We will now show that $\theta^4(z)$ is a modular form of weight 2 over the principle subgroup $\Gamma_0(4)$. First we will find its generators.

Lemma 5.1. *$\Gamma_0(4)$ is generated by* $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$

Proof. Let $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0(4)$ with $c \neq 0$.

If $c = 0$, then $a = d = \pm 1$ and then γ would be generated by $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.

Then, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix}a & *\\ c & nc+d\end{pmatrix}$. We can choose n such that $|nc+d| < |c|/2$ because c is even. Also, $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}^m = \begin{pmatrix}* & b\\ c+4md & d\end{pmatrix}$, so we can choose m such that $|c+4md| < 2d$. At each step we can strictly reduce $|c|$ or $|d|$ until $c = 0$ at which step, the matrix is some power of $\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. $\square$

We have $\theta(z) = \sum\limits_{m\in\mathbb{Z}} e^{2\pi izm^2} = \sum\limits_{m\in\mathbb{Z}} e^{2\pi im^2}\cdot e^{2\pi im^2} = \theta(z+1)$.
We will prove the identity $\theta(\frac{-1}{4z}) = \sqrt{-2iz}\ \theta(z)$ using the Poisson Summation Formula.

Lemma 5.2. ***Poisson Summation formula***
For any Schwartz function f, $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$ where $\hat{f}(x) = \int_{\mathbb{R}} f(t)e^{-2\pi ixt}dt$

Proof. Define $F(x) = \sum\limits_{n\in\mathbb{Z}} f(n+x)$. Clearly, $F(x+1) = F(x)$, so we can write F as a Fourier series.

$$\begin{aligned}
F(x) &= \sum_{k\in\mathbb{Z}}\Big(\int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)e^{-2\pi ixk}dx\Big)e^{-2\pi ikx} \\
&= \sum_{k\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}\int_0^1 f(x+n)e^{-2\pi ixk}dx\Big)e^{-2\pi ikx} \qquad \text{Since } f \text{ is Schwartz} \\
&= \sum_{k\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}\int_n^{n+1} f(x)e^{-2\pi ixk}dx\Big)e^{-2\pi ikx} \\
&= \sum_{k\in\mathbb{Z}}\Big(\int_{-\infty}^{\infty} f(x)e^{-2\pi ixk}dx\Big)e^{-2\pi ikx} \\
&= \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}
\end{aligned}$$

Substituting $x = 0$ gives us the required result. $\square$

Choosing $f(x) = e^{-\pi xt^2}$ (a Schwartz function), we get

$$\hat{f}(n) = \int_{-\infty}^{\infty} e^{-\pi tx^2 - 2\pi ixn} dx = e^{\frac{\pi n^2}{t}} \int_{-\infty}^{\infty} e^{-\pi t(x - \frac{ni}{t})^2} dx$$
$$= e^{\frac{\pi n^2}{t}} \int_{-\infty}^{\infty} e^{-\pi tx} dx \ = \frac{1}{\sqrt{t}} e^{\frac{\pi n^2}{t}}$$

Substituting $z = \frac{-t}{2i}$ we get $\theta\Big(\frac{-1}{4z}\Big) = \sum e^{\frac{\pi n^2}{t}} = \sqrt{t} \sum e^{-\pi tn^2} = \sqrt{-2iz}\ \theta(z)$

Then,

$$\begin{aligned} \theta\Big(\frac{z}{4z+1}\Big) &= \theta\Big(\frac{-1}{4(\frac{1}{4z}-1)}\Big) \\ &= \sqrt{2i(\tfrac{1}{4z}+1)}\theta(\tfrac{-1}{4z}-1) \\ &= \sqrt{2i(\tfrac{1}{4z}+1)}\theta(\tfrac{-1}{4z}) \\ &= \sqrt{2i(\tfrac{1}{4z}+1)}\sqrt{-2iz}\ \theta(z) \\ &= \sqrt{4z+1}\ \theta(z) \end{aligned}$$

Thus $\theta^4(\frac{z}{4z+1}) = (4z+1)^2\theta^4(z)$. Also, $\theta(z)$ is holomorphic at infinity because it has no Fourier coefficients for negative exponents. Thus, $\theta^4(z)$ is an element of $\mathcal{M}_2(\Gamma_0(4))$

6 The space $\mathcal{M}_2(\Gamma_0(4))$

We have shown that $\theta^4(z)$ is an element of $\mathcal{M}_2(\Gamma_0(4))$. So, it would be helpful to know more about this space. In this section, we will find the dimension of the space and find a basis. This will allow us to write $\theta^4(z)$ as a linear combination of basis elements.

6.1 Dimension of $\mathcal{M}_2(\Gamma_0(4))$

Lemma 6.1. *$dim\mathcal{M}_2(\Gamma_0(4)) = 2$*

Proof. As seen in section 3, one may look at modular forms of weight $2k$ over Γ as differential k-forms on the space $\mathbb{H}/\Gamma$. Thus, $\mathcal{M}_2(\Gamma_0(4))$ is the space of 1-forms on $\mathbb{H}/\Gamma_0(4)$ with at most simple poles at the cusps. We will find a fundamental domain for the action of $\Gamma_0(4)$ on $\mathbb{H}$. Since the translation matrix: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is in the subgroup, the fundamental domain is contained in $\{|\mathrm{Re}(z)| \le \frac{1}{2}\}$. For each matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we draw semicircles centered at $\frac{a}{c}$ with

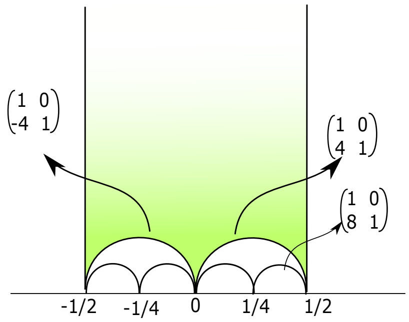

Figure 3: Fundamental domain for $\Gamma_0(4)$ acting on $\mathbb{H}$

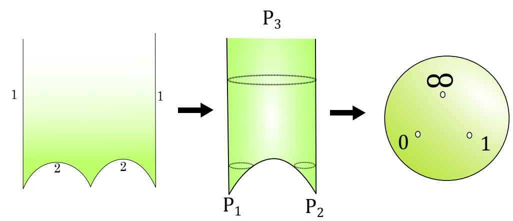

Figure 4: The space $\mathbb{H}/\Gamma_0(4)$

radius $\frac{1}{|c|}$. As seen before, a fundamental domain can be obtained by taking the region above the highest semicircles. The highest semicircles will have radius 1/4 because $c = 4N$ for some N. We find that the only matrices that contribute semicircles within the strip of width 1 are $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ and its inverse $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$. The fundamental domain is shown in Figure [3] as the shaded region. With the proper identifications, the fundamental domain becomes a sphere with 3 punctures (at 0, 1, and ∞) as shown in Figure [4].

The 1-forms on this thrice punctured sphere are allowed to have simple poles at each of the punctures. Thus the space of 1-forms is generated by 2 elements: $\frac{dz}{z}$ and $\frac{dz}{z-1}$. The 1-form $a\frac{dz}{z} + b\frac{dz}{z-1}$ has simple poles at 0 and 1. It also has a simple pole at infinity:

Set $\xi = \frac{1}{z}$. $d\xi = \frac{-1}{z^2}dz$. So the 1-form becomes: $-a\frac{d\xi}{\xi} - b\frac{d\xi}{\xi(1-\xi)}$ which has a simple pole at $\xi = 0$

Thus $\dim \mathcal{M}_2(\Gamma_0(4)) = 2$. $\square$

The next subsection finds elements of this space.

6.2 $G_2(z)$: almost an invariant of $SL(2,\mathbb{Z})$

We want a candidate for a modular form of weight 2, however, the space of modular forms of weight 2 over $SL(2,\mathbb{Z})$ was 0. We know that G_k is a modular form of weight k for $k>2$, however G_2 did not converge absolutely. To fix this, we will write G_2 in a certain way that will make it conditionally convergent.

Write $G_2(z) = \sum\limits_{c\in\mathbb{Z}} \sum\limits_{d\in\mathbb{Z}_c} \frac{1}{(cz+d)^2}$ where $\mathbb{Z}_c = \mathbb{Z}\setminus 0$ if $c=0$ and $\mathbb{Z}$ otherwise. We will show this is conditionally convergent using the following identities.

$$\pi cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z-n} + \frac{1}{z+n} \qquad .... \ (1)$$

This is true because $\pi cot(\pi z)$ has simple poles at every integer and has residue 1 at all poles.

Our second identity is:

$$\pi cot \pi z = \pi \frac{cos \pi z}{sin \pi z} = \pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = -\pi i - 2\pi i \sum_{n=0}^{\infty} e^{2\pi i n z} \qquad ...(2)$$

Differentiating $\pi cot(\pi z)$ in the first and second identity, we get $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(z+n)^2} = -4\pi^2 \sum\limits_{n=0}^{\infty} n e^{2\pi i n z}$.

$$\begin{aligned}
G_2(z) &= \sum_{c\in\mathbb{Z}} \sum_{d\in\mathbb{Z}_c} \frac{1}{(cz+d)^2} \\
&= \sum_{d\neq 0} \frac{1}{d^2} + \sum_{c>0} \sum_{d\in\mathbb{Z}} \frac{1}{(cz+d)^2} + \sum_{c<0} \sum_{d\in\mathbb{Z}} \frac{1}{(cz+d)^2} \\
&= 2\zeta(2) + 2 \sum_{c>0} \sum_{d\in\mathbb{Z}} \frac{1}{(cz+d)^2} \\
&= 2\zeta(2) + 2 \sum_{c>0} -4\pi^2 \sum_{d=0}^{\infty} d e^{2\pi i d c z} \\
&= \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z} \qquad \text{where } \sigma(n) = \sum_{d|n} d
\end{aligned}$$

We have $\left| \sum\limits_{c>0} \sum\limits_{d=0}^{\infty} d e^{2\pi i d c z} \right| \leq C \sum\limits_{c>0}^{\infty} \frac{1}{|(1-e^{2\pi i z c})^2|} \leq C' \sum\limits_{c>0}^{\infty} |e^{-4\pi i c z}| = \frac{C''}{1-|e^{-4\pi i z}|}$ which makes the series absolutely convergent. Thus we have written $G_2(z)$ as an absolutely convergent series. It is also useful to note that the Fourier coefficients have the term $\sum\limits_{d|n} 8d$ which shows up in the formula in Jacobi's four square theorem.

This definition of $G_2(z)$ makes it almost invariant of $SL(2,\mathbb{Z})$. It is clear that

$$G_2(z+1) = 2\zeta(2) - 8\pi^2 \sum_{m>0} \sum_{n=0}^{\infty} n e^{2\pi i n m (z+1)} = 2\zeta(2) - 8\pi^2 \sum_{m>0} \sum_{n=0}^{\infty} n e^{2\pi i n m z} = G_2(z)$$

.

Lemma 6.2. $G_2\left[\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right]_2(z) = G_2(z) - \frac{2\pi i}{z}$

Proof. First note that

$$\begin{aligned}
G_2\left[\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right]_2(z) &= z^{-2}G_2(\frac{-1}{z}) \\
&= z^{-2}\sum_{c\in\mathbb{Z}}\sum_{d\in\mathbb{Z}_c}\frac{1}{(-c/z+d)^2} \\
&= \sum_{c\in\mathbb{Z}}\sum_{d\in\mathbb{Z}_c}\frac{1}{(-c+dz)^2} \\
&= \sum_{d\in\mathbb{Z}}\sum_{c\in\mathbb{Z}_d}\frac{1}{(cz-d)^2} \\
&= \sum_{d\in\mathbb{Z}}\sum_{c\in\mathbb{Z}_d}\frac{1}{(cz+d)^2} \\
&= \sum_{d\neq 0}\frac{1}{d^2} + \sum_{d\in\mathbb{Z}}\sum_{c\neq 0}\frac{1}{(cz+d)^2}
\end{aligned}$$

Next, observe that using the telescopic series

$$\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)(cz+d+1)} = 0$$

We have that

$$\begin{aligned}
G_2(z) &= \frac{\pi^2}{3} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)^2} - \sum_{c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)(cz+d+1)} \\
&= \frac{\pi^2}{3} + \sum_{c\neq 0}\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)^2(cz+d+1)}
\end{aligned}$$

This double sum is on the order of $\sum_{d,c}\frac{1}{(cz+d)^3}$, and therefore converges absolutely. Rearranging the terms, we have

$$G_2(z) = \frac{\pi^2}{3} + \sum_{d\in\mathbb{Z}}\sum_{c\neq 0}\frac{1}{(cz+d)^2} - \frac{1}{(cz+d)(cz+d+1)} \quad = z^{-2}G_2(\frac{-1}{z}) - \sum_{d\in\mathbb{Z}}\sum_{c\neq 0}\frac{1}{(cz+d)(cz+d+1)}$$

Hence, to finish proving the claim, it suffices to show that

$$-\lim_{N\to\infty}\sum_{d=-N}^{N-1}\sum_{c\neq 0}\frac{1}{(cz+d)(cz+d+1)} = 2\pi i/z$$

Note that for N fixed, this sum converges absolutely. So reversing the orders of the summations gives us,

$$-\sum_{d=-N}^{N-1}\sum_{c\neq 0}\frac{1}{(cz+d)(cz+d+1)} = -\sum_{c\neq 0}\frac{1}{cz-N} + \sum_{c\neq 0}\frac{1}{cz+N}$$

Recall the cotangent identity:

$$\pi cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

Therefore,

$$-\sum_{c\neq 0}\frac{1}{cz-N} + \sum_{c\neq 0}\frac{1}{cz+N} = \sum_{c\neq 0}\frac{1/z}{N/z-c} + \frac{1/z}{N/z+c} = (1/z)2\pi cot(\pi N/z) - \frac{2z}{N}$$

Finally using the above,

$$-\lim_{N\to\infty}\sum_{d=-N}^{N-1}\sum_{c\neq 0}\frac{1}{(cz+d)(cz+d+1)} = \lim_{N\to\infty}\frac{2\pi cot(\pi N/z)}{z} = \frac{2\pi}{z}\lim_{N\to\infty} i\frac{e^{2\pi i N/z}+1}{e^{2\pi i N/z}-1} = 2\pi i/z$$

□

Lemma 6.3. *$(G_2[\gamma]_2)(z) = G_2(\gamma) - \frac{2\pi ic}{cz+d}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.*

Proof. We have shown the lemma holds for the generators of $SL(2,\mathbb{Z})$. If we can show the lemma holds under multiplication and inversion, then we are done.

The inverse of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the negative of itself, and the inverse of the translation matrix is another translation matrix, so the lemma holds for inverses.

Write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\eta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Using the general fact proven earlier that $f[\gamma]_2[\eta]_2 = f[\gamma.\eta]_2$, we would like to show that $f[\gamma.\eta]_2(z) = f(z) - \frac{2\pi i(ce+dg)}{(ce+dg)z+cf+dh}$. Indeed we have,

$$\begin{aligned} f[\gamma]_2[\eta]_2(z) &= (f(z) - \frac{2\pi ic}{cz+d})[\eta]_2 \\ &= f(z) - \frac{2\pi ig}{gz+h} - (gz+h)^{-2}\frac{2\pi ic}{c\frac{ez+f}{gz+h}+d} \\ &= f(z) - \frac{2\pi ic - 2\pi ig(c(ez+f)+d(gz+h))}{(gz+h)(c(ez+f)+d(gz+h))} \end{aligned}$$

Therefore to show multiplicativity, we need

$$\frac{2\pi ic + 2\pi ig(c(ez+f)+d(gz+h))}{(gz+h)(c(ez+f)+d(gz+h))} = \frac{2\pi i(ce+dg)}{(ce+dg)z+cf+dh}$$

Clearing the denominator, we only need to show that

$$2\pi ic + 2\pi ig(c(ez+f)+d(gz+h)) = 2\pi i(ce+dg)(gz+h)$$

After expanding and cancelling terms we have that it suffices to show

$$2\pi i(c + gcf) = 2\pi iceh$$

which holds because η has determinant 1. $\square$

6.3 A basis for $\mathcal{M}_2(\Gamma_0(4))$

We noticed that G_2 is almost invariant of $SL(2, \mathbb{Z})$ and we exploit that to define a function that is invariant of $\Gamma_0(N)$. Define $G_{2,N} = G_2(z) - NG_2(Nz)$. We will show that this is weakly modular over $\Gamma_0(N)$

Let $\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$ and $\gamma' = \begin{pmatrix} a & Nb \\ c & d \end{pmatrix}$. Thus $N\gamma(z) = \gamma'(Nz)$

$$\begin{aligned} G_{2,N}[\gamma]_2(z) &= (Ncz + d)^{-2}(G_2(\gamma(z)) - NG_2(N\gamma(z))) \\ &= G_2(z) - \frac{2\pi i Nc}{Ncz + d} - (c(Nz) + d)^{-2}(NG_2(N\gamma(z))) \\ &= G_2(z) - \frac{2\pi i Nc}{Ncz + d} - (c(Nz) + d)^{-2}(NG_2(\gamma'(Nz))) \\ &= G_2(z) - \frac{2\pi i Nc}{Ncz + d} - \left(N\left(G_2(Nz) - \frac{2\pi i c}{c(Nz) + d}\right)\right) \\ &= G_2(z) - NG_2(Nz) \end{aligned}$$

Thus $G_{2,N}$ is weakly modular over $\Gamma_0(N)$. Specifically $G_{2,4}$ is weakly modular over $\Gamma_0(4)$ and $G_{2,2}$ is weakly modular over $\Gamma_0(2)$. However, $\Gamma_0(2) \supseteq \Gamma_0(4)$, so $G_{2,2}$ is weakly modular over $\Gamma_0(4)$ as well. If we can show $G_{2,4}$ and $G_{2,2}$ satisfy the second condition for a modular form ($f[\gamma]_k$ being holomorphic at ∞), we will have a basis for the 2 dimensional vector space $\mathcal{M}_2(\Gamma_0(4))$.

We know that $G_{2,2}$ and $G_{2,4}$ are holomorphic at ∞ by looking at their Fourier expansion, but we also need $G_{2,N}[\gamma]_2$ to be holomorphic at ∞ for any $\gamma \in SL(2, \mathbb{Z})$. The following theorem gives us a way to prove that as long as the Fourier coefficients are bounded by a polynomial.

Theorem 6.4. *Let $f : \mathbb{H} \to \mathbb{C}$ be weakly modular with respect to Γ, a congruence subgroup of level N. If there exist positive constants C, r, such that the Fourier expansion of f satisfies $f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z/N}$ with $a_n \leq Cn^r$ for all $n > 0$, then*

$$|f(z)| \leq C_0 + C\left(\int_0^\infty t^r e^{-2\pi t y/N} dt\right) + \frac{C_1}{y^r}$$

Furthermore if a weakly modular function satisfies the above condition of Fourier coefficients, it is a modular form with respect to Γ.

Proof. We have $|f(z)| \leq |a_0| + \sum_{n>0} Cn^r e^{-2\pi i n y/N}$.

Consider the function $g(t) = t^r e^{-2\pi t y/N}$. $g'(t) > 0$ when $t \in (0, \frac{rN}{2\pi y})$ and $g'(t) < 0$ when $t > \frac{rN}{2\pi y}$

Calling $k = \left\lfloor \frac{rN}{2\pi y} \right\rfloor$, we get $\sum\limits_{1}^{k-1} n^r e^{-2\pi i n y} < \int\limits_0^k t^r e^{-2\pi t y/N} dt$

and $\sum\limits_{k+2}^{\infty} n^r e^{-2\pi i n y} < \int\limits_k^\infty t^r e^{-2\pi t y/N} dt$

$$\text{Thus, } |f(z)| \leq |a_0| + C\left(k^r e^{-2\pi k y/N} + (k+1)^r e^{-2\pi(k+1)y/N} + \sum_1^{k-1} n^r e^{-2\pi i n y} + \sum_{k+2}^{\infty} n^r e^{-2\pi i n y}\right)$$

$$\leq C_0 + \frac{C_1}{y^r} + C\left(\int\limits_0^\infty t^r e^{-2\pi t y/N} dt\right)$$

because $e^{-2\pi k y/N} \approx e^{-r}$ and $k^r \approx (\frac{rN}{2\pi y})^r$

For all $\gamma \in SL(2,\mathbb{Z})$, we need $f[\gamma]_k$ to be holomorphic at ∞. $f[\gamma]_k$ is invariant under $\gamma^{-1}\Gamma\gamma$ and hence has a Laurent expansion: $f[\gamma]_k(Z) = \sum_{n\in\mathbb{Z}} b_n e^{2\pi i n z/N}$.

We have $C\left(\int\limits_0^\infty t^r e^{-2\pi t y/N} dt\right) = \frac{C}{y^{r+1}}\left(\int\limits_0^\infty t^r e^{-2\pi t/N} dt\right) = \frac{C_2}{y^{r+1}}$

As $Im(z) \to \infty$,
$$\begin{aligned} |f[\gamma]_k(z)| &= |(cz+d)^{-k} f(\gamma(z))| \\ &\leq \left|(cz+d)^{-k}\left(C_0 + \tfrac{C_1}{(Im(\gamma(z)))^r} + \tfrac{C_2}{(Im(\gamma(z)))^{r+1}}\right)\right| \\ &= \left|(cz+d)^{-k}\left(C_0 + \tfrac{C_1(cz+d)^{2r}}{(Im(z))^r} + \tfrac{C_2(cz+d)^{2r+2}}{(Im(z))^{r+1}}\right)\right| \\ &= O(y^{r+1-k}) \qquad \text{since we can assume } r > 0 \end{aligned}$$

Thus, $\lim\limits_{z\to\infty} |f[\gamma]_k(z) e^{2\pi i z/N}| = \lim\limits_{y\to\infty} O(y^{r+1-k}) e^{-2\pi y/N} = 0$. This guarantees that $b_n = 0$ for all $n < 0$ giving us that $f[\gamma]_k$ is holomorphic at infinity. □

Now, we look at the fourier expansion of $G_{2,2}, G_{2,4}$.

$$\begin{aligned} G_{2,2}(z) &= G_2 - 2G_2(2z) \\ &= 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i z n} - 2\left(2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) e^{4\pi i z n}\right) \\ &= -\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty}\left(\sum_{d|n, d\notin 2\mathbb{Z}} d\right) e^{2\pi i z n} \end{aligned}$$

Similarly, $G_{2,4}(z) = -\pi^2 - 8\pi^2 \sum\limits_{n=1}^{\infty}\left(\sum\limits_{d|n, d\notin 4\mathbb{Z}} d\right) e^{2\pi i z n}$.

The fourier coefficients are bounded by $8\pi^2\sigma(n) \leq 8\pi^2 n^2$ and we can apply the theorem. Hence, $G_{2,2}, G_{2,4} \in \mathcal{M}_2(\Gamma_0(4))$ and are a basis.

7 The formula for the sum of four squares

Now that we know the space $\mathcal{M}_2(\Gamma_0(4))$ has dimension 2 and a basis for the space is $G_{2,2}$ and $G_{2,4}$, we should be able to write $\theta^4(z)$ as a linear combination of these two functions. We have

$$\begin{aligned} \theta^4(z) &= aG_{2,2} + bG_{2,4} \\ \implies 1 + 8e^{2\pi iz} + ... &= \frac{-a\pi^2}{3}(1 + 24e^{2\pi iz} + ...) - b\pi^2(1 + 8e^{2\pi iz} + ...) \end{aligned}$$

Comparing the constant term and thecoefficients of the $e^{2\pi iz}$ term, we get, $a = 0$ and $b = \frac{-1}{\pi^2}$ giving us $\theta^4(z) = \frac{-1}{\pi^2} G_{2,4} = \sum\limits_{n\in\mathbb{Z}} \left(8 \sum\limits_{d|n, d\notin 4\mathbb{Z}} d\right) e^{2\pi izn}$. Thus we have proved $r_4(n) = 8 \sum\limits_{d|n, d\notin 4\mathbb{Z}} d$

8 References

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