Arithmetic Kleinian groups

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Contents

1 Introduction

Kleinian groups are discrete subgroups of $PSL(2, \mathbb{C})$. It's well known that the isomorphism classes of Kleinian groups Γ that don't contain elliptics are in one to one correspondence with hyperbolic 3-manifolds given by $\mathbb{H}^3/\Gamma$. We are interested in those groups that have finite *covolume*: (i.e.) $\mathbb{H}^3/\Gamma$ has finite volume, and arithmetic Kleinian groups are an important family of finite covolume Kleinian groups.

Arithmetic Kleinian groups come from quaternion algebras as we will see in Section 3 and the necessary preliminaries are developed in Section 2.

While arithmetic Kleinian groups have interesting properties on their own, one can ask about the 3-manifold they produce. One would hope that the extra arithmetic structure makes it easier to study arithmetic 3-manifolds, and we look at some interesting questions in Section 4. The first is about arithmetic Kleinian groups that produce knot or link complements. The second is whether volumes of arithmetic Kleinian groups satisfies any nice properties.

Note: Most of the above material is obtained from [2], but a more condensed version of the next 2 sections can be found at [1].

2 Preliminaries

This section will develop necessary preliminaries about number fields, quaternion algebras, trace fields and commensurable groups.

2.1 Number fields

Let k be a field extension of $\mathbb{Q}$ of degree n. There exists an element $t \in k$ such that its minimal polynomial in $\mathbb{Q}[x]$ has degree n. Let $t_1, t_2, \ldots t_n$ be the roots of this polynomial. Each t_i induces an embedding σ_i of k into $\mathbb{C}$ induced by $\sigma_i(t) = t_i$. σ_i is called a *real* place if $\sigma_i(k) \subset \mathbb{R}$. Otherwise, these embeddings come in conjugate pairs, and each such pair is called a *complex* place. Clearly $n = r_1 + 2r_2$ where r_1 is the number of real places and r_2, the number of complex places.

2.2 Quaternion algebras

A quaternion algebra A over a field k of char $\neq 2$, is a 4 dimentional vector space over k with basis $1, i, j, k$ such that 1 is a multiplicative identity, $i^2 = a1$, $j^2 = b1$ and $ij = -ji = k$ for some

$a, b \in k^\times$. We will write a as $\left(\frac{a,b}{k}\right)$.

Example 2.1. The usual Hamiltonian quaternions $\mathcal{H} = \left(\frac{-1,-1}{k}\right)$.

Another example is $M_2(k) = \left(\frac{1,1}{k}\right)$ where $i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $j = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

Some easy properties to check are

- $\left(\frac{a,b}{k}\right) \cong \left(\frac{ax^2,by^2}{k}\right)$ for any $a, b, x, y \in k^\times$.
- For any field extension L of k, we have $\left(\frac{a,b}{k}\right) \otimes_k L \cong \left(\frac{a,b}{L}\right)$
- For any embedding $\sigma : k \to L$, we have $\left(\frac{a,b}{k}\right) \otimes_{\sigma(k)} L \cong \left(\frac{\sigma(a),\sigma(b)}{L}\right)$

Over the complex numbers, every element is a square, and thus by the first property, every quaternion algebra is equivalent to $M_2(\mathbb{C})$. On the other hand, quaternion algebras over the reals can either be $M_2(\mathbb{R})$ or $\mathcal{H}$.

Definition 2.2. Let k be a field extension of $\mathbb{Q}$ of degree n. A quaternion algebra A over k is said to be *ramified* at a real place $\sigma : k \to \mathbb{C}$ if $A \otimes_\sigma \mathbb{R} \cong \mathcal{H}$, and unramified otherwise.

Definition 2.3. Let A be a quaternion algebra over a number field k and let R be the ring of integers in k. An *order* $\mathcal{O} \subset A$ is an R-lattice in A that contains 1.

We can define norms and traces for quaternion algebras as follows: Let $A_0 \subset A$ be the "pure quaternions" (those spanned by i, j, k). Any $x \in A$ can be uniquely written as $a1 + \alpha$ with $a \in k$ and $\alpha \in A_0$. Define the conjugate of x as: $\overline{x} = a1 - \alpha$. Then, the norm of x is: $n(x) = x\overline{x}$ and the trace $tr(x) = x + \overline{x}$. Later, we will be interested in elements of an order of A with norm 1.

2.3 Commensurable groups

Definition 2.4. 2 groups Γ and Γ' are said to be *commensurable* if there are subgroups $\Delta \subseteq \Gamma, \Delta' \subseteq \Gamma'$ such that Γ/Δ and Γ'/Δ' are finite index and $\Delta \cong \Delta'$.

In some situations, like in the case of Kleinian groups, we can pick $\Delta = \Delta' = \Gamma \cap \Gamma'$ and redefine commensurability to mean that $\Gamma \cap \Gamma'$ is finite index in each of the groups. Two hyperbolic manifolds $\mathbb{H}^3/\Gamma$ and $\mathbb{H}^3/\Gamma'$ are said to be commensurable if they have diffeomorphic finite coverings. This is equivalent to Γ being commensurable to some conjugate of Γ'. Studying commensurability classes of Kleinian groups could tell us more about hyperbolic 3-manifolds. For instance, Γ and Γ' are commensurable finite covolume Kleinian groups, then, $\frac{\mathrm{Vol}(\mathbb{H}^3/\Gamma')}{\mathrm{Vol}(\mathbb{H}^3/\Gamma)} = \frac{[\Gamma/\Gamma\cap\Gamma']}{[\Gamma':\Gamma\cap\Gamma']}$

2.4 Trace fields

Given a Kleinian group Γ with finite covolume, we define its trace field: $\mathbb{Q}(tr\Gamma)$ as the field obtained by adjoining the traces of elements in Γ with $\mathbb{Q}$. This will be a finite extension of $\mathbb{Q}$. However, it is not a commensurability class invariant.

Define $\Gamma^{(2)} = \{\gamma^2 | \gamma \in \Gamma\}$. Denote the trace field of $\Gamma^{(2)}$ as $k\Gamma$. Remarkably, this is a commensurability class invariant of Γ.

3 Arithmetic groups

3.1 Definition

Definition 3.1. Let k be a number field with exactly 1 complex place and let A be a quaternion algebra over k which is ramified at all real places. Let ρ be a k-embedding of A into $M_2(\mathbb{C})$ and let $\mathcal{O}$ be an order of A. Then a subgroup Γ of $PSL_2(\mathbb{C})$ is an arithmetic Kleinian group if it is commensurable with some $P\rho(\mathcal{O}^1)$ where $\mathcal{O}^1$ are the elements of $\mathcal{O}$ of norm 1.

Remark 3.2. The field k and the algebra A in the definition are not so mysterious. In fact $k = k\Gamma$, the invariant trace field, and $\rho(A) = A\Gamma = \{\sum a_i\gamma_i | a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)}\}$, the invariant quaternion algebra.

3.2 Examples

Bianchi Groups are an important class of examples. They are the groups: $PSL(2, \mathcal{O}_d)$ where $\mathcal{O}_d = \mathbb{Z}[\sqrt{-d}]$. We notice that $k = k\Gamma = \mathbb{Q}(\sqrt{-d})$ which has only 1 complex place and no real places, so $A = M_2(k)$ is vacuously ramified at all real places. We have $\mathcal{O} = M_2(k)$ is an order of A, ρ is the identity and $P\rho(\mathcal{O}^1)$ is precisely $PSL(2, \mathcal{O}_d)$.

From definition 3.1 of Arithmetic groups, it is clear that if a Kleinian group is commensurable to an arithmetic kleinian group, it is arithmetic as well. The following is an important classification theorem of commensurability classes of arithmetic kleinian groups:

Theorem 3.3. *Any non co-compact arithmetic Kleinian group is commensurable to a Bianchi group.*

In [5], Riley showed that that the fundamental group of the figure-8 knot is commensurable with $PSL(2, \mathcal{O}_3)$ and hence arithmetic. The knot group for the figure-8 is $\{x, y \mid wx = yw, w = x^{-1}yxy^{-1}\}$. We can obtain a representation of this group in $PSL_2(\mathbb{C})$ by $x \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and

$y \mapsto \left(\begin{smallmatrix} 1 & 0 \\ -w & 1 \end{smallmatrix}\right)$ where $w = \frac{-1+\sqrt{-3}}{2}$. One needs to check this is a homomorphism (which is not so easy) and then shows that the subgroup generated by these two matrices is finite index in $PSL(2, \mathcal{O}_3)$

Remark 3.4. The meridian x and the longitude y both map to parabolics. As we will see in the following section, such representations play an important role in identifying arithmetic knots.

4 Questions

4.1 Arithmetic knots and links

Question. *What knots or links are arithmetic? (i.e.) for what arithmetic kleinian groups Γ is $\mathbb{H}^3/\Gamma$ a knot or link complement?*

In 1979, Riley in [5] showed that several link complements are arithmetic, and constructed infinite families of arithmetic links. Some examples are the two bridge link (10/3), the Whitehead link, and the Borromean rings. An infinite family can be obtained by taking branched cyclic covers over an unknotted component of the Borromean rings (From Chapter 9 in [2]).

We know that the figure-8 knot is arithmetic (see section 3.2). However no other such knots were found. In 1991, Reid proved in [3], that the only arithmetic knot was the figure 8. The following paragraphs outline some key ideas to his proof.

The first is the observation that the knot group of the figure-8 knot is commensurable with $PSL(2, \mathcal{O}_3)$. Furthermore, its knot group has an "excellent" representation in $PSL(2, \mathcal{O}_3)$. Although this has a technical definition, one can understand it as the meridian in the fundamental group mapping to a parabolic element in $PSL(2, \mathcal{O}_3)$. We know that any arithmetic knot group will be commensurable to some Bianchi group by Theorem 3.3. Reid further proves that all arithmetic knots should have excellent representation in some $PSL(2, \mathcal{O}_d)$ by looking at the trace field and using the proposition: a non-cocompact Kleinian group Γ of finite covolume is derived from a quaternion algebra if and only if $\mathrm{tr}\Gamma \subset \mathcal{O}_d$ for some d.

The second idea is the following connection between the class number h_d of $PSL(2, \mathcal{O}_d)$ and the number of cusps of $\mathbb{H}^3/PSL(2, \mathcal{O}_d)$. It turns out that h_d simply counts the counts the conjugacy classes of maximal parabolic subgroups, which is precisely the number of cusps. Thus, if a knot group has excellent representation in $PSL(2, \mathcal{O}_d)$ then, $h_d = 1$ because a knot has only 1 cusp. A little more work limits the search to $d = 1, 2, 3, 7, 11, 19$.

Then it is a matter of careful elimination to show that for $d = 1, 2, 7, 9, 11$, any arithmetic group Γ will have elliptic elements. Once $d = 3$, one can show it has to be the figure 8.

4.2 Volumes of Arithmetic manifolds

The following is a theorem of Borel (Theorem 11.1.3 in [2]):

Theorem 4.1. *Let k be a number field with exactly one complex place, A be a quaternion algebra over k ramified at all real places and $\mathcal{O}$ be a maximal order in A. Then if ρ is a k-embedding of A to $M_2(\mathbb{C})$ then,*

$$\mathit{Volume}(\mathbb{H}^3/P\rho(\mathcal{O}^1)) = \frac{4\pi^2|\Delta_k|^{3/2}\zeta_k(2)\prod_{\mathcal{P}|\Delta(A)}(N(\mathcal{P})-1)}{(4\pi^2)^{[k:\mathbb{Q}]}}$$

where Δ_k is the discriminant of the field k (or of the polynomial that determines k), $\Delta(A)$ is the discriminant of A, ζ_k is the Dedekind zeta function of k, $N(\mathcal{P})$ is the norm of the prime ideal $\mathcal{P}$.

The formula guarantees that there only finitely many arithmetic 3-manifolds below any volume threshold (Theorem 11.2.1 in [2]) and hence there should be a minimum volume manifold. The Weeks manifold is the smallest volume arithmetic 3-manifold, whereas the figure 8 knot has the least volume among orientable cusped manifolds. These were also conjectured to be the minimal volume hyperbolic manifolds (that are not necessarily arithmetic) and in 2008, the work of Gabai, Meyerhoff, and Milley showed that this was indeed the case [6].

References

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