# Heegaard Floer Homology and Knots

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## Contents

# 1 Introduction.

In this paper we give an expository account of an invariant for 3-manifolds called Heegaard Floer homology, introduced by Peter Ozsváth and Zoltán Szabó in [7] and [6]. Heegaard Floer homology is related to other Floer homology invariants for 3-manifolds; in particular, it is conjectured to be equivalent to Seiberg-Witten monopole Floer homology [6]. It also has strong ties with knot theory. In fact, we will discuss a related invariant called knot Floer homology. This invariant was introduced independently in [13] and [5]. It is defined for general null-homologous knots in 3-manifolds, although in this paper we will restrict attention to knots in $S^3$.

In Section 2.1 through Section 2.4, we give the definition of Heegaard Floer homology and discuss some properties which will be relevant in later sections. In Section 2.5, we define knot Floer homology and state some of its properties. Finally, in Section 2.6, we discuss holomorphic triangles, cobordisms, and induced maps on Heegaard Floer homology.

One important property of Heegaard Floer homology is that one can compute it for sufficiently large knot surgeries using knot Floer homology. In Section 3.1, we state some general theorems along these lines. We then focus on the corresponding calculations in knot Floer homology. From Section 3.2 to Section 3.4, we outline an approach to computing the knot Floer chain complex for a general knot in $S^3$, based on techniques using Fox calculus which were introduced in [13]. We then consider three specific examples in Section 3.5. Finally, we prove the general theorems relating knot Floer homology to large surgeries in Section 3.6.

In the third major section of the paper, Section 4, we apply the surgery exact triangle in Heegaard Floer homology to prove a result of Gordon and Luecke [2] that knots are determined by their complements. After outlining the proof in Section 4.1, we state the exact triangle in Section 4.3 and prove it in

Section 4.4. The rest of Section 4 carries out the proof of the result of Gordon and Luecke; in Section 4.7, we take a detour to define Heegaard Floer homology with twisted coefficients.

Clearly, surgeries on knots in $S^3$ will play an important role in much of what is to follow, so for convenience we fix the notation before beginning. Knots will always be oriented, unless otherwise specified. If $K$ is an oriented knot in $S^3$ and $\mathrm{nb}\, K$ is a tubular neighborhood of $K$, then a *meridian* for $K$ is an embedded circle $\mu$ in $\partial(\mathrm{nb}\, K)$ which bounds a disk in $\mathrm{nb}\, K$. Its homology class in $H_1(\partial(\mathrm{nb}\, K))$ is unique up to sign; we will use the term "meridian" to refer either to the circle or to its homology class $[\mu]$.

Any homology class $\lambda$ in $\partial(\mathrm{nb}\, K)$ which has intersection $\pm 1$ with $\mu$ is called a *longitude* for $K$. Given a longitude $\lambda$ for $K$, the other possible longitudes for $K$ are obtained from $\lambda$ by adding integer multiples of $[\mu]$, so they are indexed by $\mathbb{Z}$. A knot equipped with a choice of longitude is called a *framed knot*, and the longitude is sometimes called a *framing*. One can perform $\lambda$-framed surgery on $K$ by removing $\mathrm{int}(\mathrm{nb}\, K)$ from $S^3$, gluing on a two-handle along $\lambda$, and capping off the result with a 3-ball. The manifold obtained by this process will be denoted $K_\lambda$.

Since $K$ is a knot in $S^3$, it has a canonical framing, the *Seifert framing*, defined as follows: choose a Seifert surface $F$ of $K$. Then, as long as $\mathrm{nb}\, K$ is small enough, $F$ intersects $\partial(\mathrm{nb}\, K)$ in a circle, and this circle defines a longitude $\lambda_{Seif}$ of $K$. For knots in arbitrary three-manifolds, this definition might not be independent of $F$, but for knots in $S^3$ it is.

Now that $K$ has a preferred longitude $\lambda_{Seif}$, we can index all longitudes for $K$ by $\mathbb{Z}$: let $\lambda_0 := \lambda_{Seif}$ and let $\lambda_p := \lambda_0 + p[\mu]$. This definition depends on the sign we choose for the meridian; we want to fix this choice in some way. To do so, consider the surgery $K_{\lambda_{Seif}}$ (henceforth denoted $K_0$). In this manifold, we can cap off $F$ to obtain a closed surface $\hat{F}$. Since $K$ is oriented, $F$ and hence $\hat{F}$ inherit orientations from $K$. We may also view $[\mu]$ as an element of $H_1(K_0)$. To fix the sign of $[\mu]$, we require that $\langle PD[\mu], [\hat{F}]\rangle = +1$ rather than $-1$.

Now we may unambiguously refer to the longitudes $\lambda_p$ for $p \in \mathbb{Z}$. The surgery $K_{\lambda_p}$ will be denoted simply by $K_p$, the *p-surgery* on $S^3$ along $K$ (or just the *p-surgery*, when $K$ is implied).

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# 2 Definitions of Heegaard Floer homology and knot Floer homology.

In this section we will define the Heegaard Floer homology of a three-manifold $Y$, denoted $\widehat{HF}(Y)$, as well as the variants $HF^-(Y)$, $HF^+(Y)$, and $HF^\infty(Y)$. We will also define the knot Floer homology $\widehat{HFK}(S^3, K)$ of a knot in $S^3$. While in this paper we will only consider knots in $S^3$, knot Floer homology can be defined for a null-homologous knot in any closed three-manifold. We will give proofs when appropriate, but many will be omitted for reasons of space. The exposition will roughly follow [9].

Unless otherwise specified, three-manifolds will always be taken to be closed and oriented, although we will often mention these conditions explicitly as well to avoid confusion.

## 2.1 Heegaard decompositions and Heegaard diagrams.

To define the Heegaard Floer homology of a three-manifold $Y$, we start by choosing a suitable Heegaard diagram for $Y$. Thus, we first need to discuss what this means.

### 2.1.1 Heegaard decompositions of three-manifolds.

**Definition 2.1.** Let $Y$ be a closed oriented 3-manifold. A genus-$g$ *Heegaard decomposition* of $Y$ is an identification of $Y$ with a manifold of the form $U_1 \cup_\phi U_2$, where $U_1$ and $U_2$ are both handlebodies of a common genus $g$. Here, $\phi : \partial U_1 \to \partial U_2$ is a homeomorphism of the genus-$g$ surface $\Sigma_g = \partial H_g$ specifying how the two copies $U_1$ and $U_2$ of $H_g$ are to be glued.

It is a basic fact that three-manifolds always admit Heegaard decompositions:

**Proposition 2.2.** *Let $Y$ be a closed oriented 3-manifold. Then $Y$ has a Heegaard decomposition.*

*Proof.* One way to obtain a Heegaard decomposition is by taking a handle decomposition of $Y$, i.e. by writing $Y$ = 0-handle $\cup$ 1-handle $\cup$ 2-handles $\cup$ 3-handle. Then the union of the 0-handle and the 1-handles is a genus-$g$ handlebody for some $g$; call the handlebody $U_1$. Dually, the union of the 2-handles and the 3-handle is also a handlebody $U_2$ of genus $g'$ for some $g'$. However, since $\partial(U_1) = \partial(\overline{Y \setminus U_1}) = \partial(U_2)$, we must have $g = g'$.

Another way to obtain a Heegaard decomposition of $Y$ is by triangulating $Y$, fattening the vertices and edges, and taking $U_1$ to be the union of the fattened vertices and edges. It is clear that the closure of the complement of $U_1$ is again a handlebody, since it consists of the fattened barycentres of the 3-simplices connected by handles corresponding to the barycentres of the 2-simplices. The same argument as above shows that these two handlebodies must have the same genus. □

### 2.1.2 Heegaard diagrams.

Let $U_1 \cup_\phi U_2$ be a Heegaard decomposition of a three-manifold $Y$. By introducing a redundancy in our description of this decomposition, we obtain an easy way to visualize it. Namely, rather than viewing the boundaries of $U_1$ and $U_2$ as glued together with a single homeomorphism, we will consider both $\partial U_1$ and $\partial U_2$ as glued to an abstract genus-$g$ surface $\Sigma$. The redundancy is that we need two homeomorphisms $\phi_1$ and $\phi_2$ in this picture, rather than just one.

The homeomorphisms $\phi_i : \partial U_i \to \Sigma$ each specify a system of $g$ circles in $\Sigma$, as follows. Let $\gamma_1, \dots, \gamma_g$ be disjoint circles in $\partial U_1$ representing those independent homology classes in $\partial U_1$ which are zero (i.e. bound disks) in $U_1$. Then $\phi_1(\gamma_1), \dots, \phi_g(\gamma_g)$ are disjoint circles in $\Sigma$ representing independent homology classes. Denote these circles by $\alpha_1, \dots, \alpha_g$. We have similar circles $\beta_1, \dots, \beta_g$ coming from $U_2$ and $\phi_2$.

Motivated by this discussion, we make the following definition:

**Definition 2.3.** Let $\Sigma$ be a genus-$g$ surface.

(a) A set of $g$ circles $\alpha_1, \dots, \alpha_g$ in $\Sigma$ is called a *system of attaching circles* in $\Sigma$ if the $\alpha_i$ are disjoint and determine independent homology classes in $H_1(\Sigma)$.

(b) (b) A genus-$g$ *Heegaard diagram* is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\Sigma$ is an oriented surface of genus $g$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_g)$ are two systems of attaching circles in $\Sigma$.

The above discussion shows that given a Heegaard decomposition of a three-manifold, we can make some choices and get a Heegaard diagram. Up until now, however, we have not made use of the given orientation on $Y$, and we have not given any reason for choosing one orientation on $\Sigma$ over the other. As one might expect, we want these two things to be compatible. More precisely, we want to choose the orientation on $\Sigma$ such that the oriented three-manifold $Y'$ associated to $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ (as defined below) is equal to $Y$ rather than $\overline{Y}$.

Given an oriented surface $\Sigma$ and two sets of attaching circles $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_g)$, we want to construct an associated oriented 3-manifold. We can start with $\Sigma \times [0, 1]$, with its orientation induced from $\Sigma$ and the standard orientation on $[0, 1]$. View the $\alpha$ circles as living in $\Sigma \times \{1\}$ and the $\beta$ circles as living in $\Sigma \times \{0\}$. Now attach a two-handle along each $\alpha_i$ and $\beta_j$. After adding all the two-handles, we cap off the result with two three-handles. This procedure specifies a way of gluing two handlebodies to $\Sigma$ along its boundary to obtain a three-manifold $Y$. The orientation on $Y$ is determined by the orientation on the open set $\Sigma \times (1/3, 2/3)$, which was specified before any gluing was performed. We say that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for $Y$ or "represents" $Y$.

*Remark* 2.4. For future use, it will be important to note that if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ represents $Y$, then $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ both represent $\overline{Y}$, whereas $(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$ represents $Y$.

Clearly, if we start with a three-manifold $Y$, pick a Heegaard diagram by taking a decomposition of $Y$ as described above, and then construct the three-manifold associated to the diagram by gluing 2-handles along the attaching circles, we recover $Y$. Thus, we have (informally) proved the following proposition:

**Proposition 2.5.** *Let $Y$ be a closed oriented three-manifold. There exists a Heegaard diagram for $Y$ (in fact, there exist many).*

It will sometimes be useful to have a Morse-theoretic approach to Heegaard diagrams. The following proposition will summarize what we need:

**Proposition 2.6.** *Let $Y$ be a closed oriented 3-manifold.*

(a) *There exists a self-indexing Morse function $f$ on $Y$ with unique critical points of index 0 and 3 and with $g$ critical points each of index 1 and 2, for some positive integer $g$.*

(b) *If $f$ is such a Morse function, then $f$ determines a Heegaard diagram of $Y$ as follows: define the Heegaard surface $\Sigma$ to be $f^{-1}(3/2)$, a surface of genus $g$. Label the index 1 critical points of $f$ as $x_1, \ldots, x_g$ and the index 2 critical points as $y_1, \ldots, y_g$. For $1 \leq i \leq g$, let $\alpha_i$ be the set of points in $\Sigma$ which flow to $x_i$ under $-\nabla f$, and let $\beta_i$ be the set of points which flow to $y_i$ under $\nabla f$. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram representing $Y$.*

(c) *Every Heegaard diagram for $Y$ arises from a Morse function as in (b).*

*Remark* 2.7. A Morse function $f$ on $Y$ as in Proposition 2.6 induces a handle decomposition of $Y$ and hence a Heegaard decomposition of $Y$ via Proposition 2.2. This decomposition agrees with the decomposition associated to the Heegaard diagram of Proposition 2.6(b); in both cases, $U_1$ is $f^{-1}([0, 3/2])$ and $U_2$ is $f^{-1}([3/2, 3])$. The $\alpha$ curves are the belt spheres of the 1-handles, and the $\beta$ curves are the attaching spheres of the 2-handles. This fact will be relevant in Section 2.4.2.

## 2.2 Other preliminaries.

### 2.2.1 Symmetric products.

Given a three-manifold $Y$, we want to define its Heegaard Floer homology as, roughly, the Lagrangian Floer homology of a certain manifold and submanifolds associated to a Heegaard diagram for $Y$. We define the manifold and submanifolds now.

**Definition 2.8.** Let $\Sigma$ be a genus-$g$ surface.

(a) The $g^{th}$ *symmetric product* of $\Sigma$, denoted $\operatorname{Sym}^g \Sigma$, is the quotient of $\Sigma \times \cdots \times \Sigma$ ($g$ times) by the natural action of the symmetric group $S_g$ by permuting coordinates. In other words, it is the set of unordered $g$-tuples of points in $\Sigma$, with repetitions allowed.

(b) Let $\Sigma$ be as above and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ be a set of $g$ attaching circles in $\Sigma$. The torus $\mathbb{T}_{\boldsymbol{\alpha}}$ associated to $\boldsymbol{\alpha}$ is $(\alpha_1 \times \cdots \times \alpha_g)/S_g$.

*Remark* 2.9. Clearly, the $k^{th}$ symmetric power of any space $X$ can be defined in the same way. However, if $X$ is a genus-$g$ surface $\Sigma$, then $\operatorname{Sym}^k \Sigma$ is actually a manifold of dimension $2k$. The reason is that $\operatorname{Sym}^k \Sigma$ locally looks like an open set of unordered tuples of $k$ complex numbers, and thus can be seen as an open set of monic polynomials of degree $k$ over $\mathbb{C}$ by the fundamental theorem of algebra. The monic polynomials over $\mathbb{C}$ are homeomorphic to $\mathbb{C}^k$. Also, for our purposes, $k = g$ will the the only relevant power.

*Remark* 2.10. Since attaching circles must be disjoint, no two distinct points of $\alpha_1 \times \cdots \times \alpha_g$ are in the same orbit of $S_g$. Thus, $\mathbb{T}_{\boldsymbol{\alpha}}$ is homeomorphic to $\alpha_1 \times \cdots \times \alpha_g \simeq (S^1)^g$, so we are justified in calling it a torus. It is a real submanifold of $\operatorname{Sym}^g \Sigma$.

Now say we have a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. We then have a $2g$-dimensional manifold $\operatorname{Sym}^g \Sigma$ and two $g$-dimensional submanifolds $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. In Section 2.3, we will proceed as in Lagrangian Floer homology and define a group associated to $(\operatorname{Sym}^g \Sigma, \mathbb{T}_{\boldsymbol{\alpha}}, \mathbb{T}_{\boldsymbol{\beta}})$. The result will be the Heegaard Floer homology of our original manifold $Y$.

### 2.2.2 Homotopy classes of Whitney disks and moduli spaces of holomorphic representatives.

As in Lagrangian Floer homology, the generators of the chain complex for Heegaard Floer homology will be intersection points $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, and the differentials will come from holomorphic mappings of disks into $\operatorname{Sym}^g \Sigma$ which "start" and "end" at intersection points. We define the relevant spaces of these Whitney disks here. We will not discuss the analytic details; they can be found in [7].

**Definition 2.11.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. A *Whitney disk* connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ is a map $\phi : D^2 \to \operatorname{Sym}^g \Sigma$ such that $\phi(-i) = \boldsymbol{x}$, $\phi(i) = \boldsymbol{y}$, $\phi(z) \in \mathbb{T}_{\boldsymbol{\alpha}}$ for $|z| = 1$ and $\operatorname{Re} z \geq 0$, and $\phi(x) \in \mathbb{T}_{\boldsymbol{\beta}}$ for $|z| = 1$ and $\operatorname{Re} z \leq 0$. Define $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ to be the space of homotopy classes of Whitney disks connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. We will continue to write $\phi$ for an element of $\pi_2(\boldsymbol{x}, \boldsymbol{y})$.

If we choose a complex structure on $\Sigma$, we get an induced complex structure on $\operatorname{Sym}^g \Sigma$, so we can talk about holomorphic maps of $D^2$ into $\operatorname{Sym}^g \Sigma$. Let $\phi$ be a homotopy class in $\pi_2(\boldsymbol{x}, \boldsymbol{y})$; certain representatives of $\phi$ may be holomorphic. Similarly, if we choose a different almost-complex structure $J$ on $\operatorname{Sym}^g \Sigma$, we can talk about pseudoholomorphic representatives of $\phi$. In fact, when we say "holomorphic," we will almost always mean "pseudoholomorphic with respect to a suitably perturbed almost-complex structure," but we will not work at this level of detail.

Given a holomorphic representative $\phi_0$ of $\phi$, we can obtain others by reparametrization as follows. Map $D^2$ conformally to the strip $0 < \operatorname{Re} z < 1$; then $\phi_0$ is a holomorphic map from the strip into $\operatorname{Sym}^g \Sigma$, and we can obtain another by precomposing $\phi_0$ with a translation by $it$ for any $t \in \mathbb{R}$. The resulting map is homotopic to $\phi_0$, so it is a holomorphic representative of the homotopy class $\phi$. In this way, we have an action of $\mathbb{R}$ on the set of holomorphic representatives of $\phi$.

**Definition 2.12.** Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and let $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$.

(a) $\mathcal{M}(\phi)$ is the set of holomorphic representatives of $\phi$ with respect to some almost-complex structure $J$ on $\operatorname{Sym}^g \Sigma$.

(b) $\overline{\mathcal{M}}(\phi) := \mathcal{M}(\phi)/\mathbb{R}$, where the action of $\mathbb{R}$ on $\mathcal{M}(\phi)$ by reparametrization is that described in the above paragraph.

For suitable perturbations of the almost-complex structure on $\operatorname{Sym}^g \Sigma$, these moduli spaces $\mathcal{M}(\phi)$ and $\overline{\mathcal{M}}(\phi)$ are manifolds in a natural way. Furthermore, there exists a mapping $\mu : \pi_2(\boldsymbol{x}, \boldsymbol{y}) \to \mathbb{Z}$ called the *Maslov index*. For $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, one expects $\mu(\phi)$ to be the dimension of $\mathcal{M}(\phi)$, and for a suitable perturbation of the almost-complex structure, this is the case. In fact, we have the following theorem:

**Theorem 2.13.** *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram. Choose a complex structure on $\Sigma$. For suitable perturbations of the induced almost-complex structure on $\operatorname{Sym}^g \Sigma$, we have the following: for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$,*

*(a) $\mathcal{M}(\phi)$ is an orientable manifold of dimension $\mu(\phi)$, and*

*(b) $\overline{\mathcal{M}}(\phi)$ is an orientable manifold of dimension $\mu(\phi) - 1$, and*

*(c) If $\mu(\phi) = 1$, then $\overline{\mathcal{M}}(\phi)$ is compact.*

To obtain orientations on the moduli spaces, we would need to make more choices; see Definition 3.11 of [7]. We will not worry about this particular detail; we will assume that all moduli spaces come with orientations.

### 2.2.3 $\mathrm{Spin}^c$ structures.

For some $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, the set $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ is empty. In fact, there is a simple necessary and sufficient condition for $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ to be nonempty. Choose a path $\sigma$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ in $\mathbb{T}_{\boldsymbol{\alpha}}$ and a path $\tau$ from $\boldsymbol{y}$ to $\boldsymbol{x}$ in $\mathbb{T}_{\boldsymbol{\beta}}$. After homotoping to avoid intersecting the diagonal if necessary, we may lift $\sigma$ and $\tau$ to $\Sigma \times \cdots \times \Sigma$. But a path in $\Sigma \times \cdots \times \Sigma$ corresponds to $g$ paths in $\Sigma$; hence $\sigma$ and $\tau$ give us two sets of $g$ paths $\{\sigma_1, \ldots, \sigma_g\}$ and $\{\tau_1, \ldots, \tau_g\}$ in $\Sigma$. Write $c_{\boldsymbol{x},\boldsymbol{y}} = \sum \sigma_i + \sum \tau_i$; then $c_{\boldsymbol{x},\boldsymbol{y}}$ is a cycle. Note that $c_{\boldsymbol{x},\boldsymbol{y}}$ depends on some arbitrary choices of liftings as well as on $\boldsymbol{x}$ and $\boldsymbol{y}$.

There is a simple way to construct $c_{\boldsymbol{x},\boldsymbol{y}}$ which does not involve $\operatorname{Sym}^g \Sigma$. Suppose $\boldsymbol{x} = \{x_1, \ldots, x_g\}$ and $\boldsymbol{y} = \{y_1, \ldots, y_g\}$. Start at $x_1$; it lies on some curve $\alpha_i$. Follow $\alpha_i$ in any direction until reaching a point $y_j$, and add the chosen path to the chain $c_{\boldsymbol{x},\boldsymbol{y}}$. The point $y_j$, in turn, lies on some curve $\beta_k$; follow $\beta_k$ in any direction until reaching a point $x_l$, and add the path to $c_{\boldsymbol{x},\boldsymbol{y}}$. Take another $\alpha$ curve to a point of $\boldsymbol{y}$, then another $\beta$ curve back to a point of $\boldsymbol{x}$, etc. At some point, one returns to $x_1$. If all intersection points have been exhausted, then $c_{\boldsymbol{x},\boldsymbol{y}}$ is complete. If not, start with an unused point of $\boldsymbol{x}$ and continue the process until all intersection points are used up. Again, choices have been made, and $c_{\boldsymbol{x},\boldsymbol{y}}$ is not unique.

It will, most likely, be the case that $c_{\boldsymbol{x},\boldsymbol{y}} \neq 0$ in $H_1(\Sigma)$. However, we may use the inclusion $\Sigma \hookrightarrow Y$ to view $c_{\boldsymbol{x},\boldsymbol{y}}$ as an element of $H_1(Y)$. The following basic proposition asserts that including $c_{\boldsymbol{x},\boldsymbol{y}}$ in $Y$ amounts to looking at $c_{\boldsymbol{x},\boldsymbol{y}}$ modulo the $\alpha$ and $\beta$ curves:

**Proposition 2.14.** *Let $\iota : \Sigma \hookrightarrow Y$ be the inclusion map. The induced homomorphism*

$$\frac{H_1(\Sigma)}{\langle [\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g] \rangle} \xrightarrow{\iota_*} H_1(Y)$$

*is an isomorphism.*

Now we can state the necessary and sufficient condition for $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ to be nonempty:

**Proposition 2.15.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Then $\iota_* c_{\boldsymbol{x},\boldsymbol{y}}$ is independent of the choices made in its definition; we will call it $\epsilon(\boldsymbol{x}, \boldsymbol{y})$. Furthermore, $\pi_2(\boldsymbol{x}, \boldsymbol{y}) = \emptyset$ if and only if $\epsilon(\boldsymbol{x}, \boldsymbol{y}) \neq 0$ in $H_1(Y)$, which happens if and only if $c_{\boldsymbol{x},\boldsymbol{y}} \in H_1(\Sigma)$ is in the span of the $\alpha$ and $\beta$ curves.*

It is clear that for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, we can use $c_{\boldsymbol{x},\boldsymbol{y}} + c_{\boldsymbol{y},\boldsymbol{z}}$ to compute $c_{\boldsymbol{x},\boldsymbol{z}}$, and hence $\epsilon(\boldsymbol{x}, \boldsymbol{y}) + \epsilon(\boldsymbol{y}, \boldsymbol{z}) = \epsilon(\boldsymbol{x}, \boldsymbol{z})$. Thus, $\epsilon$ gives the elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ a relative grading by $H_1(Y) = H^2(Y)$. We would like to lift this $\epsilon$-grading to an absolute grading in a natural way. The solution will be to absolutely grade elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ according to $\mathrm{Spin}^c$ structures on $Y$, which form an affine space modelled on $H^2(Y)$. Differences in the $\mathrm{Spin}^c$ grading will correspond to the $\epsilon$-differences defined above.

For three-manifolds, the most useful definition of $\mathrm{Spin}^c$-structures will be the following (due to Turaev [14]):

**Definition 2.16.** Let $Y$ be a three-manifold.

(a) Two nowhere-vanishing vector fields $v_1$ and $v_2$ on $Y$ are said to be *homologous* if there exists a 3-ball $B$ inside $Y$ such that $v_1$ and $v_2$ are homotopic after restriction to $Y \setminus B$.

(b) A $\mathrm{Spin}^c$-structure on $Y$ is a homology class of nowhere-vanishing vector fields on $Y$.

(c) If $\mathfrak{s}$ is a $\mathrm{Spin}^c$-structure on $Y$ represented by a nowhere-vanishing vector field $v$, its conjugate $\mathrm{Spin}^c$-structure $\bar{\mathfrak{s}}$ is the one represented by $-v$.

*Remark* 2.17. It is a well-known fact that all closed oriented three-manifolds are parallelizable. Thus, they admit nowhere-vanishing vector fields and hence $\mathrm{Spin}^c$-structures.

In fact, let $Y$ be a closed orientable three-manifold; since $Y$ is parallelizable, nowhere-vanishing vector fields on $Y$ correspond (after choosing a trivialization) to maps of $Y$ into $\mathbb{R}^3 \setminus \{0\}$. Suppose $v$ is such a map. Since $\mathbb{R}^3 \setminus \{0\} \sim S^2$, its second cohomology group is generated by some fixed class $\omega$. Pulling back $\omega$ by $v$, we get an element of $H^2(Y)$. In fact, it can be shown that the association $v \mapsto v^*\omega$ is defined on the level of $\mathrm{Spin}^c$ structures and gives a bijection between $\mathrm{Spin}^c(Y)$ and $H^2(Y)$. If $\phi : TY \simeq Y \times \mathbb{R}^3$ is the trivialization we chose, we will call this bijection $F_\phi : \mathrm{Spin}^c(Y) \to H^2(Y)$.

$F_\phi$ is not independent of the trivialization $\phi$. However, given any two $\mathrm{Spin}^c$ structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on $Y$, one can consider the element $F_\phi(\mathfrak{s}_2) - F_\phi(\mathfrak{s}_1)$ in $H^2(Y)$. It turns out that this difference is independent of $\phi$:

**Proposition 2.18.** *If $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathrm{Spin}^c(Y)$, then $F_\phi(\mathfrak{s}_2) - F_\phi(\mathfrak{s}_1) \in H^2(Y)$ is independent of the trivialization $\phi$. We will write $F_\phi(\mathfrak{s}_2) - F_\phi(\mathfrak{s}_1)$ simply as $\mathfrak{s}_2 - \mathfrak{s}_1$.*

This proposition tells us that $H^2(Y)$ acts freely and transitively on $\mathrm{Spin}^c(Y)$ independently of the trivialization of $TY$. In other words, $\mathrm{Spin}^c(Y)$ is naturally an affine space over $H^2(Y)$.

We now define a $\mathrm{Spin}^c(Y)$-grading on elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Choose a basepoint $z \in \Sigma$ disjoint from the $\alpha$ and $\beta$ curves. By Proposition 2.6(c), we can pick a Morse function $f$ inducing the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Suppose $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ corresponds to the unordered set $\{x_1, \ldots, x_g\}$ of points in $\Sigma$. Each $x_i$ lies on a gradient flow of $-f$ from an index 2 critical point to an index 1 critical point. Similarly, $z$ lies on a flow from the index 3 critical point to the index 0 critical point. These flows trace out $g+1$ segments in $Y$. Define $Y_0$ to be $Y$ minus a tubular neighborhood of each of the $g+1$ segments.

On $Y_0$, the vector field $\nabla f$ is nonvanishing. Furthermore, $Y \setminus Y_0$ is a disjoint union of 3-balls. One 3-ball contains an index 3 and an index $-3$ critical point of $f$, while the others contain an index 1 and an index $-1$ critical point each. Thus, on the boundary of each 3-ball, the index of $\nabla f$ is zero. Hence we can extend $\nabla f$ to a nonvanishing vector field on all of $Y$, thereby obtaining a $\mathrm{Spin}^c$ structure on $Y$. Define $\mathfrak{s}_z(\boldsymbol{x})$ to be this $\mathrm{Spin}^c$ structure. The below proposition indicates how the map $\mathfrak{s}_z$ behaves. It also describes the dependence of $\mathfrak{s}_z$ on the basepoint $z$. To state this second part of the proposition, we need a definition. Inside the handlebody $U_1$ formed by attaching the $\alpha$ circles to $\Sigma$, $\alpha_i$ becomes the belt sphere of a 1-handle. There is a corresponding "dual" circle in $U_1$, which we can push back up into $\Sigma$ to obtain a curve $\alpha_i^*$. Then $\alpha_i^*$ intersects $\alpha_i$ once and has zero intersection with the other $\alpha$ curves. Let $[\alpha_i^*]$ denote the homology class of $\alpha_i^*$ in $H_1(Y)$, and let $PD[\alpha_i^*]$ denote its Poincare dual in $H^2(Y)$.

**Proposition 2.19.** *Let $z \in \Sigma$ be a point in the complement of the $\alpha$ and $\beta$ curves. The procedure described above gives a well-defined map $\mathfrak{s}_z : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(Y)$, where the subscript $z$ indicates the dependence of this map on the basepoint $z$. It associates to $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ the $\mathrm{Spin}^c$ structure $\mathfrak{s}_z(x)$ in such a way that*

*(a) the equation*

$$\mathfrak{s}_z(\boldsymbol{y}) - \mathfrak{s}_z(\boldsymbol{x}) = PD(\epsilon(\boldsymbol{x}, \boldsymbol{y}))$$

*holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$; and*

*(b) if $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, and $z, z'$ are two basepoints connected by a small arc intersecting $\alpha_i$ once and having zero intersection with the other $\alpha$ and $\beta$ curves, then*

$$\mathfrak{s}_z(\boldsymbol{x}) - \mathfrak{s}_{z'}(\boldsymbol{x}) = \pm PD[\alpha_i^*].$$

For a proof of Proposition 2.19, see Section 2.6 of [7].

Given a basepoint $z$, we have now interpreted the relative $H^2(Y)$-grading $\epsilon$ on $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ as a natural partitioning of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ according to $\mathrm{Spin}^c$ structures on $Y$. Our construction of Heegaard Floer homology will respect this splitting; for each $\mathrm{Spin}^c$ structure $\mathfrak{s}$ on $Y$ we will define a group $\widehat{HF}(Y, \mathfrak{s})$, and we will have $\widehat{HF}(Y) := \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s})$.

Finally, the following definition will be important in the future:

**Definition 2.20.** Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. The *first Chern class* $c_1(\mathfrak{s})$ of $\mathfrak{s}$ is defined as $\mathfrak{s} - \overline{\mathfrak{s}} \in H^2(Y)$.

### 2.2.4 Basepoints.

We saw in the previous section that the map $\mathfrak{s}_z : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(Y)$ depends on a choice of basepoint $z \in \Sigma$. In fact, we will always assume that we have chosen some basepoint $z$ in $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g\}$. The basepoint is also important in the definition of the differential in the chain complex below. For this purpose, we introduce the following definition:

**Definition 2.21.** Let $z \in \Sigma$ be a basepoint in the complement of the $\alpha$ and $\beta$ curves.

(a) $V_z$ is defined as $\{z\} \times \Sigma \times \cdots \times \Sigma / S_g$, a submanifold of $\mathrm{Sym}^g \Sigma$ of dimension $2g - 2$.

(b) Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$. Then $n_z(\phi)$ is the signed intersection number between $\phi$ and $V_z$. This definition does not depend on the choice of representative of the homotopy class $\phi$, as long as a suitable (e.g. transverse to $V_z$) representative is chosen.

### 2.2.5 Domains.

In Section 2.2.3, we characterized when $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ is nonempty. When it is nonempty, we want a more concrete description of $\pi_2(\boldsymbol{x}, \boldsymbol{y})$. Unfortunately, it is hard to directly visualize maps from $D^2$ into the $2g$-dimensional manifold $\mathrm{Sym}^g \Sigma$ when $g > 1$. Luckily, we will be able to think of a Whitney disk in terms of a 2-chain in $\Sigma$ called its domain. Besides aiding visualization, the use of domains will be essential in showing that the sums in the differential of the chain complex we will define are finite.

Consider the (closures of the) components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g\}$; we will denote them by $\{\sigma_i\}$. A domain in $\Sigma$ is a formal sum of the regions $\sigma_i$ which "connects two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$." We now make this precise:

**Definition 2.22.** Let $\boldsymbol{x} = \{x_1, \ldots, x_g\}$ and $\boldsymbol{y} = \{y_1, \ldots, y_g\}$ be points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$.

(a) A *domain* connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ is a formal linear combination of the $\sigma_i$, say $D = \sum n_i \sigma_i$, such that $\partial D$ consists of $2g$ arcs, $g$ of which connect $x_j$ to $y_{\tau(j)}$ for some permutation $\tau$, and $g$ of which connect $y_j$ to $x_{\tau'(j)}$ for some other permutation $\tau'$. The set of domains connecting $\boldsymbol{x}$ to $\boldsymbol{y}$ is denoted by $\mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$.

(b) Suppose $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$. The *domain of* $\phi$, denoted $D(\phi)$, is defined as follows: choose a point $z_i$ in the interior of each $\sigma_i$. Then $D(\phi) := \sum n_i \sigma_i$, where $n_i$ is the signed intersection number of $\operatorname{im} \phi$ (for a suitable representative of $\phi$) with the submanifold $V_{z_i}$ of $\operatorname{Sym}^g \Sigma$. The coefficients $n_i$ are independent of the $z_i$ and of the representative of $\phi$.

The following result tells us that, for diagrams of genus $g > 2$, homotopy classes of Whitney disks are determined by their domains:

**Proposition 2.23.** *Suppose $g > 2$. The map $\phi \mapsto D(\phi)$ is a bijection between $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ and $\mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$. If $g = 2$, the map is a surjection. If $g = 1$, it is an injection. Furthermore, if $w \in \Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g\}$ and $n_w(D)$ denotes the coefficient of a domain $D$ on the region $\sigma_i$ containing $w$, then for $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, we have $n_w(\phi) = n_w(D(\phi))$.*

Accordingly, we redefine $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ when $g = 2$:

**Definition 2.24.** Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram of genus 2. Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. From now on, $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ will mean the set {homotopy classes of Whitney disks connecting $\boldsymbol{x}$ to $\boldsymbol{y}$} modulo the relation $\phi_1 \sim \phi_2$ if $D(\phi_1) = D(\phi_2)$. With this definition, the map $D$ is now a bijection between $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ and $\mathcal{D}(\boldsymbol{x}, \boldsymbol{y})$.

It is not hard to determine all the possible domains connecting $\boldsymbol{x}$ to $\boldsymbol{y}$. Namely, suppose $D = \sum n_i \sigma_i$ is such a domain. Then, since the sum of the regions $\sigma_i$ is a generator for $H_2(\Sigma)$, we see that $\sum (n_i + 1)\sigma_i$ is also a domain connecting $\boldsymbol{x}$ to $\boldsymbol{y}$. Informally, we have "added $\Sigma$" to $D$. In general, for any $j \in \mathbb{Z}$, the domain $\sum (n_i + j)\sigma_i$ connects $\boldsymbol{x}$ to $\boldsymbol{y}$.

When $H_2(Y) = 0$, these domains are the only domains connecting $\boldsymbol{x}$ to $\boldsymbol{y}$. In general, however, elements of $H_2(Y)$ correspond to *periodic domains*, i.e. domains $P$ such that $n_z(P) = 0$ and $\partial P = \sum k_i \alpha_i + \sum l_j \beta_j$ for some $k_i, l_j$. If $D$ is a domain connecting $\boldsymbol{x}$ and $\boldsymbol{y}$, and $P$ is a periodic domain, then $D + P$ is also a domain connecting $\boldsymbol{x}$ to $\boldsymbol{y}$. We have now found all such domains:

**Proposition 2.25.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Then there is a one-to-one correspondence between $D(\boldsymbol{x}, \boldsymbol{y})$ and $\mathbb{Z} \times H_2(Y)$. Hence, combining this result with Proposition 2.23, there is a one-to-one correspondence between $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ and $\mathbb{Z} \times H_2(Y)$.*

**Corollary 2.26.** *Suppose $b_1(Y) = 0$, i.e. $Y$ is a rational homology three-sphere. Let $z$ be a basepoint in the Heegaard diagram for $Y$, as in Section 2.2.4. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $\epsilon(\boldsymbol{x}, \boldsymbol{y}) = 0$. Then there is a unique $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ with $n_z(\phi) = 0$.*

*Proof.* Suppose $z \in \sigma_{i_0}$. We want to show that there exists a unique domain $D'$ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ with a coefficient of zero on $\sigma_{i_0}$. Let $D = \sum n_i \sigma_i$ be any domain connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. By Proposition 2.25, the unique choice for $D'$ is $D' = \sum (n_i - n_{i_0})\sigma_i$. □

There is a concrete algorithm to determine the domain of some element $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, given $\boldsymbol{x}$ and $\boldsymbol{y}$. Consider the cycle $c_{\boldsymbol{x},\boldsymbol{y}}$ constructed in Section 2.2.3. If it represents a nontrivial element of $\frac{H_1(\Sigma)}{\langle [\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g] \rangle} = H_1(Y)$, then $\epsilon(\boldsymbol{x}, \boldsymbol{y}) \neq 0$, so $\pi_2(\boldsymbol{x}, \boldsymbol{y}) = \emptyset$. If, on the other hand, $c_{\boldsymbol{x},\boldsymbol{y}}$ is zero in this group, then it lies in the span of the $\alpha$ and $\beta$ curves, so by adding some multiples of these curves, we obtain a chain $c'_{\boldsymbol{x},\boldsymbol{y}}$ which is zero in $H_1(\Sigma)$. The regions $\sigma_i$ specify a cell decomposition of $\Sigma$ and hence a cellular chain complex computing $H_*(\Sigma)$. Thus, we may express $c'_{\boldsymbol{x},\boldsymbol{y}}$ as the boundary of some domain $D$ formed as a linear combination of the regions $\sigma_i$. Then $D$ is the domain $D(\phi)$ of some $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, by the correspondence between domains and Whitney disks, and the possible choices for $D$ arising from this algorithm are precisely the domains of disks in $\pi_2(\boldsymbol{x}, \boldsymbol{y})$.

In the case where $Y$ is a rational homology three-sphere, i.e. $H_2(Y) = 0$, there is a unique such $D$ satisfying $n_z(D) = 0$. Again, we would like a way of constructing this $D$ without dealing with $\operatorname{Sym}^g \Sigma$. Consider $\partial D$, which equals $c'_{\boldsymbol{x},\boldsymbol{y}}$ in the notation of the above paragraph. For each region $\sigma_i$, we want a way of computing the coefficient of $D$ on $\sigma_i$. Choose a point $w_i$ in the interior of $\sigma_i$ and a path $\gamma_i$ from $z$

to $w_i$ which intersects $c'_{\boldsymbol{x},\boldsymbol{y}}$ transversely. Start at $z$ and label the region in which $z$ lies with the coefficient 0. Now begin walking along $\gamma_i$. When leaving one region and entering another, set the coefficient of the new region to be that of the old region if there is no component of $c'_{\boldsymbol{x},\boldsymbol{y}}$ along the boundary. If, on the other hand, $c'_{\boldsymbol{x},\boldsymbol{y}}$ is traversed from right to left (according to the orientation of $c'_{\boldsymbol{x},\boldsymbol{y}}$) with multiplicity $n$, take the new coefficient to be $n$ plus the old coefficient. If $c'_{\boldsymbol{x},\boldsymbol{y}}$ is traversed from left to right with multiplicity $n$, take the new coefficient to be $-n$ plus the old coefficient. In this way, we eventually label the region $\sigma_i$, and we can perform this procedure with any $\sigma_i$. The resulting coefficients are the coefficients of $D$ on the regions $\sigma_i$.

## 2.3 The definition of Heegaard Floer homology.

### 2.3.1 The definition of $\widehat{HF}$.

We can now proceed to define the chain complex giving rise to the most basic form of Heegaard Floer homology, denoted $\widehat{HF}$. We first make the definition for rational homology three-spheres:

**Definition 2.27.** Let $Y$ be a closed oriented three-manifold with $b_1(Y) = 0$. Let $\mathfrak{s}$ be a Spin$^c$ structure on $Y$. Choose a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $Y$ and a basepoint $z$ in $\Sigma$ disjoint from the $\alpha$ and $\beta$ curves. Choose a complex structure on $\Sigma$ and a suitable perturbation of the induced complex structure on $\mathrm{Sym}^g \Sigma$, as in Theorem 2.13 (where $g$ is the genus of $\Sigma$).

(a) As a group, $\widehat{CF}(Y)$ is defined as $\mathbb{Z}\langle \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \rangle$, the free Abelian group on the points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$.

(b) Let $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. The differential of $\boldsymbol{x}$ is defined to be

$$\partial x = \sum_{\{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, \phi \in \pi_2(\boldsymbol{x},\boldsymbol{y}) | \mu(\phi)=1, n_z(\phi)=0\}} \# \overline{M}(\phi) \cdot y.$$

**Theorem 2.28.** *The differential satisfies $\partial^2 = 0$, so we can define $\widehat{HF}(Y) = \frac{\ker \partial}{\operatorname{im} \partial}$.*

Several remarks are in order. First of all, our assumption that $b_1(Y) = 0$ implies that for any two generators $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, there exists at most one $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ with $n_z(\phi) = 0$ (see Corollary 2.26). Thus, there is no problem with the finiteness of the sum.

Second of all, if $\epsilon(\boldsymbol{x}, \boldsymbol{y}) \neq 0$, i.e. (by Proposition 2.19) if the Spin$^c$ structures $\mathfrak{s}(\boldsymbol{x})$ and $\mathfrak{s}(\boldsymbol{y})$ are different, then the $y$-component of $\partial x$ vanishes. Thus, $\widehat{CF}(Y)$ splits up as a sum of complexes according to Spin$^c$ structures:

**Definition 2.29.** Let $Y$ be as above and let $\mathfrak{s}$ be a Spin$^c$ structure on $Y$. Then $\widehat{CF}(Y, \mathfrak{s}) := \mathbb{Z}\langle \{\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} | \mathfrak{s}(\boldsymbol{x}) = \mathfrak{s}\} \rangle$. By what has been said, $\widehat{CF}(Y, \mathfrak{s})$ is a subcomplex of $\widehat{CF}(Y)$, and we have $\widehat{CF}(Y) = \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{CF}(Y, \mathfrak{s})$.

As a third remark, the notation $\widehat{CF}(Y)$ is slightly misleading, since the chain complex depends on the choices we made. The following theorem, though, tells us that the choices only matter up to homotopy equivalence.

**Theorem 2.30.** *Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Making different choices in the above definitions leads to chain homotopy equivalent complexes $\widehat{CF}(Y, \mathfrak{s})$.*

One peculiar aspect of our construction of $\widehat{HF}$ is that we introduced no homological grading on $\widehat{CF}$. By analogy with ordinary homology, we might have expected a homological grading such that the differential $\partial$ lowered homological degree by 1. For general 3-manifolds, the best we can do in this direction is a relative $\mathbb{Z}/2$ grading on Heegaard Floer homology. Since we are dealing in this section with rational homology three-spheres, however, we can do a little better: we can introduce a relative $\mathbb{Z}$-grading on each $\widehat{CF}(Y, \mathfrak{s})$.

**Definition 2.31.** For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $\epsilon(\boldsymbol{x}, \boldsymbol{y}) = 0$, define $F(\boldsymbol{x}, \boldsymbol{y}) = \mu(\phi) - 2n_z(\phi)$, where $\phi$ is any class in $\pi_2(\boldsymbol{x}, \boldsymbol{y})$. Since $b_1(Y) = 0$, this definition is independent of the choice of $\phi$.

In the definition of the differential $\partial$, we only count disks $\phi$ which satisfy $\mu(\phi) = 1$ and $n_z(\phi) = 0$, so $\partial$ lowers degree by one as expected. Note also that this relative grading permits a definition of the Euler characteristic (up to an overall sign) of each group $\widehat{HF}(Y, \mathfrak{s})$ in the usual way.

This grading does not lift to an absolute $\mathbb{Z}$-grading in a natural way; instead, it lifts naturally to an absolute $\mathbb{Q}$-grading in which the grading differences are all integers. We will assume the existence of this natural $\mathbb{Q}$-lift, but we will omit its definition due to space considerations.

For integer homology three-spheres, the absolute $\mathbb{Q}$-gradings turn out to live in $\mathbb{Z}$, so in this very special case we do have an absolute $\mathbb{Z}$-grading. In particular, for any Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for $S^3$, the complex $\widehat{CF}(S^3)$ computed using this diagram is absolutely $\mathbb{Z}$-graded. The Heegaard Floer homology of $S^3$ is localized in degree 0: $\widehat{HF}_0(S^3) = \mathbb{Z}$, and $\widehat{HF}_i(S^3) = 0$ for all $i \neq 0$.

### 2.3.2 Other variants: $HF^\infty, HF^-$, and $HF^+$.

In the above definition of $\widehat{HF}$, we ensured finiteness of the differential by requiring $n_z(\phi) = 0$ for any contributing $\phi$. There is another way to ensure finiteness of each coefficient which allows $\phi$ to have $n_z(\phi) \neq 0$. One simply counts $\phi$ with different values of $n_z(\phi)$ as coefficients of different formal generators in the expression for $\partial x$. Along these lines, we have the following definition:

**Definition 2.32.** Suppose $Y$ is a three-manifold, and make appropriate choices as in Definition 2.27.

(a) As a group, $CF^\infty(Y) := \mathbb{Z}\langle(\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}) \times \mathbb{Z}\rangle$, the free Abelian group on generators $[\boldsymbol{x}, i]$ where $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $i \in \mathbb{Z}$.

(b) Let $[\boldsymbol{x}, i] \in (\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}) \times \mathbb{Z}$. The differential of $[\boldsymbol{x}, i]$ in the complex $CF^\infty(Y)$ is

$$\partial[\boldsymbol{x}, i] := \sum_{\{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, \phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) | \mu(\phi)=1\}} \# \overline{M}(\phi) \cdot [\boldsymbol{y}, i - n_z(\phi)].$$

(c) $\partial^2 = 0$ holds as before, and we define $HF^\infty(Y) = \frac{\ker \partial}{\operatorname{im} \partial}$. Also as before, we have a splitting of $CF^\infty(Y)$ into subcomplexes according to Spin$^c$ structures: $CF^\infty(Y) = \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} CF^\infty(Y, \mathfrak{s})$. There is a corresponding splitting of the homology: $HF^\infty(Y) = \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} HF^\infty(Y, \mathfrak{s})$.

Finiteness of $\partial$ follows from the fact that for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, there is at most one $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ with $\mu(\phi) = 1$. For a proof, see Proposition 2.15 and Lemma 3.3 of [7]. Note that this argument could equally well have been applied to $\widehat{CF}$.

We can define a relative $\mathbb{Z}$-grading as before: if $[\boldsymbol{x}, i]$ and $[\boldsymbol{y}, j]$ are two generators for $CF^\infty(Y)$, then $F([\boldsymbol{x}, i], [\boldsymbol{y}, j]) := F(\boldsymbol{x}, \boldsymbol{y}) + 2i - 2j$. The differential $\partial$ still decreases the degree by one.

There is an obvious automorphism of $CF^\infty(Y)$, denoted by $U$, which sends the generator $[\boldsymbol{x}, i]$ to $[\boldsymbol{x}, i-1]$. This automorphism decreases the relative homological grading by 2. Thus, $CF^\infty(Y)$ is naturally a module over $\mathbb{Z}[U, U^{-1}]$ (where here $U$ is a formal variable acting on $CF^\infty(Y)$ via the automorphism $U$). In fact, for any $Y$ with $b_1(Y) = 0$, and for any $\mathfrak{s}$, $HF^\infty(Y, \mathfrak{s})$ is just the trivial module $\mathbb{Z}[U, U^{-1}]$ over $\mathbb{Z}[U, U^{-1}]$; see Section 10 of [6]. Nevertheless, subcomplexes and quotients of $CF^\infty(Y, \mathfrak{s})$ will have interesting homology.

**Definition 2.33.** Let $Y$ be as in Definition 2.27, with appropriate choices. Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

(a) $CF^-(Y, \mathfrak{s})$ is defined to be the subcomplex of $CF^\infty(Y, \mathfrak{s})$ spanned by those generators $[\boldsymbol{x}, i]$ with $i \leq 0$. It is naturally a module over $\mathbb{Z}[U]$.

(b) $CF^+(Y, \mathfrak{s})$ is defined to be the quotient of $CF^\infty(Y, \mathfrak{s})$ by $CF^-(Y, \mathfrak{s})$. It is naturally a module over $\frac{\mathbb{Z}[U, U^{-1}]}{(U \cdot \mathbb{Z}[U])}$.

*Remark* 2.34. Elements of $\frac{\mathbb{Z}[U, U^{-1}]}{(U \cdot \mathbb{Z}[U])}$ are just polynomials in $U^{-1}$, so we will sometimes write this ring as $\mathbb{Z}[U^{-1}]$. The reason we defined it as a quotient of $\mathbb{Z}[U, U^{-1}]$ rather than directly as $\mathbb{Z}[U^{-1}]$ was to emphasize the fact that it has a natural action of $U$ (as well as of $U^{-1}$).

Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Given a choice of Heegaard diagram, there is an obvious short exact sequence of complexes $0 \to CF^-(Y, \mathfrak{s}) \to CF^\infty(Y, \mathfrak{s}) \to CF^+(Y, \mathfrak{s}) \to 0$. As usual, such a short exact sequence induces a long exact sequence in homology. The homology sequence does not depend on the choice of Heegaard diagram:

**Theorem 2.35.** *There exists a long exact sequence*

$$\cdots \to HF^-(Y,\mathfrak{s}) \to HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s}) \to \cdots$$

*which depends only on $Y$ and $\mathfrak{s}$ (up to isomorphism of sequences).*

*Proof.* This result is Theorem 2.1 of [8]. □

*Remark* 2.36. Because there are no absolute $\mathbb{Z}$-gradings on the complexes, this long exact sequence is actually an exact triangle. In other words, in the statement of Theorem 2.35, the map on the far right of the sequence is the same as the map on the far left, and the sequence keeps repeating in this manner. Another exact triangle, associated to knot surgeries, will be very important in Section 4.

### 2.3.3 Three-manifolds $Y$ with $b_1(Y) > 0$.

We will often consider manifolds which are obtained as zero-surgeries on knots in $S^3$ and hence have nonzero first Betti number. Thus, we should extend our definitions to include manifolds $Y$ with $b_1(Y) > 0$. As mentioned above, we will not get a relative $\mathbb{Z}$-grading on Heegaard Floer homology in this case. A more pressing issue, though, is that our arguments for the finiteness of the differential in the complexes defined above do not hold here. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with $\epsilon(\boldsymbol{x}, \boldsymbol{y}) = 0$, there may be infinitely many distinct classes $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ with $\mu(\phi) = 1$ and $n_z(\phi) = 0$.

Proposition 2.25 illustrates this phenomenon. We have $\pi_2(\boldsymbol{x}, \boldsymbol{y}) \simeq \mathbb{Z} \times H_2(Y)$, but while requiring $n_z(\phi) = 0$ fixes the $\mathbb{Z}$-component of $\phi$, there is still an $H_2(Y)$-degree of freedom in $\phi$ coming from the addition of periodic domains. It may be the case that infinitely many of these possible $\phi$ have $\mu(\phi) = 1$.

To avoid this problem, we will not allow ourselves to choose an arbitrary Heegaard diagram for $Y$. Rather, we will restrict attention to "weakly admissible" and "strongly admissible" Heegaard diagrams, as defined below:

**Definition 2.37.** Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram.

(a) $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is *weakly admissible* if all (nonzero) periodic domains have at least one positive and at least one negative coefficient.

(b) Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is *strongly admissible* for $\mathfrak{s}$ if, for any periodic domain $D$ such that $\langle c_1(\mathfrak{s}), H(D)\rangle = 2n$, some coefficient of $D$ is greater than $n$. Here $H(D)$ denotes the element of $H_2(Y)$ corresponding to $D$ under the bijection $H_2(Y) \leftrightarrow \{$ periodic domains $\}$.

To see why weak admissibility ensures the finiteness in the differential for $\widehat{HF}$, suppose $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ admits a holomorphic representative. Then $\mathrm{im}\,\phi$ is a complex submanifold of $\mathrm{Sym}^g \Sigma$ and hence has only positive intersections with the complex submanifolds $V_w$ for any $w$. Since the coefficients in the domain $D(\phi)$ were defined to be intersection numbers with $V_w$ for various $w$, these coefficients must all be positive. But if all periodic domains $P$ have both positive and negative coefficients, then only finitely many domains of the form $D(\phi) + P$ have all positive coefficients. This means that only finitely many elements $\phi'$ of $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ with $n_z(\phi') = 0$ can possibly admit holomorphic representatives, as required.

In general, weak admissibility will suffice for $\widehat{HF}(Y)$ and $HF^+(Y)$, while $HF^-(Y,\mathfrak{s})$ and $HF^\infty(Y,\mathfrak{s})$ will require strong admissibility for $\mathfrak{s}$ in order to be well-defined.

Although there is no relative $\mathbb{Z}$-grading in this context, there is a relative $\mathbb{Z}/2$-grading. Fix orientations on the $\alpha$ and $\beta$ curves, inducing orientations on $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$. Given this choice of orientations, we define an absolute $\mathbb{Z}/2$ grading on elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. It will depend on the orientations, but the associated relative grading will not. Suppose $\boldsymbol{x}$ is an intersection points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Say $\mathrm{gr}(\boldsymbol{x}) = 1$ if, at $\boldsymbol{x}$, the basis for $T_{\boldsymbol{x}} \mathrm{Sym}^g \Sigma$ obtained by concatenating an oriented basis for $T_{\boldsymbol{x}}\mathbb{T}_{\boldsymbol{\alpha}}$ and an oriented basis for $T_{\boldsymbol{x}}\mathbb{T}_{\boldsymbol{\beta}}$ agrees with the orientation on $T_{\boldsymbol{x}} \mathrm{Sym}^g \Sigma$ coming from the orientation on $\Sigma$, i.e. if $\mathbb{T}_{\boldsymbol{\alpha}}$ intersects $\mathbb{T}_{\boldsymbol{\beta}}$ positively at $\boldsymbol{x}$. Say $\mathrm{gr}(\boldsymbol{x}) = -1$ otherwise. It is clear that changing the choice of orientations on the $\alpha$ and $\beta$ curves affects the grading of each $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ in the same way. Thus, the induced relative $\mathbb{Z}/2$ grading is independent of the choices.

Given a choice of orientations on the $\alpha$ and $\beta$ curves, we can compute $\mathrm{gr}(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ without explicit reference to $\mathrm{Sym}^g \Sigma$ as follows. The point $\boldsymbol{x}$ corresponds to $g$ points $x_1, \ldots, x_g$ in $\Sigma$, where

$x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma$. Let $\epsilon_i(\boldsymbol{x})$ be $+1$ if $\alpha_i$ intersects $\beta_{\sigma(i)}$ positively at $x_i$, and let it be $-1$ otherwise. The formula

$$\mathrm{gr}(\boldsymbol{x}) = \mathrm{sgn}(\sigma) \sum_{i=1}^{g} \epsilon_i(\boldsymbol{x}) \tag{1}$$

follows from the way in which orientations on the $\alpha$ and $\beta$ curves and on $\Sigma$ induce orientations on $\mathbb{T}_{\boldsymbol{\alpha}}$, $\mathbb{T}_{\boldsymbol{\beta}}$, and $\mathrm{Sym}^g \Sigma$.

For our purposes, this relative $\mathbb{Z}/2$ grading is important since with it, the Euler characteristic of $\widehat{HF}(Y, \mathfrak{s})$ still makes sense (up to overall sign as in the previous case) for each Spin$^c$ structure $\mathfrak{s}$.

## 2.4 Properties of Heegaard Floer homology.

We will need a few facts about Heegaard Floer homology later; they will be collected here.

### 2.4.1 Conjugation symmetry.

**Proposition 2.38.** *Let $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and let $\overline{\mathfrak{s}}$ denote its conjugate. Then $\widehat{HF}(Y, \mathfrak{s}) \simeq \widehat{HF}(Y, \overline{\mathfrak{s}})$.*

*Proof.* Choose a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $Y$. As noted in Section 2.1.2, $(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$ also represents $Y$. Since Heegaard Floer homology is independent of the choice of Heegaard diagram, either diagram may be used to compute $\widehat{HF}(Y)$. The first diagram computes $\widehat{HF}(Y)$ in the usual way. In the second, the generators and differentials are the same. However, the Spin$^c$ structures associated to intersection points $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ are different. By Proposition 2.6(c), we can pick a Morse function $f$ inducing the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$; then $-f$ induces $(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$. If $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, let $\mathfrak{s}_z(\boldsymbol{x})$ be the Spin$^c$ structure induced by $\boldsymbol{x}$ using $f$, and let $\mathfrak{s}'_z(\boldsymbol{x})$ be the one induced using $-f$ (here $z$ is a basepoint as usual). There is a finite number of balls in $Y$ such that on their complement, $\mathfrak{s}_z(\boldsymbol{x}) = \overline{\nabla f}$ and $\mathfrak{s}'_z(\boldsymbol{x}) = -\nabla f$. But $\overline{\mathfrak{s}_z(\boldsymbol{x})}$ is also represented by $-\nabla f$ on this complement. Hence $\mathfrak{s}'_z(\boldsymbol{x})$ and $\overline{\mathfrak{s}_z(\boldsymbol{x})}$ are represented by homologous nowhere-vanishing vector fields, so they are equal. We can conclude that $\widehat{HF}(Y, \mathfrak{s}_z(\boldsymbol{x})) = \widehat{HF}(Y, \mathfrak{s}'_z(\boldsymbol{x})) = \widehat{HF}(Y, \overline{\mathfrak{s}_z(\boldsymbol{x})})$. But all Spin$^c$ structures $\mathfrak{s}$ on $Y$ for which $\widehat{HF}(Y, \mathfrak{s}) \neq 0$ are of the form $\mathfrak{s}_z(\boldsymbol{x})$ for some $\boldsymbol{x}$. Thus, $\widehat{HF}(Y, \mathfrak{s}) = \widehat{HF}(Y, \overline{\mathfrak{s}})$ for all $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. □

### 2.4.2 The Euler characteristic of $\widehat{HF}$.

**Proposition 2.39.** *Suppose $Y$ is a three-manifold and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Then*

$$\chi(\widehat{HF}(Y, \mathfrak{s})) = \begin{cases} \pm 1 & \text{if } H_1(Y) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We will prove the result in two steps. The first will be to show that $\chi(\widehat{HF}(Y)) = \#(H_1(Y))$ if $H_1(Y)$ is finite and $\chi(\widehat{HF}(Y)) = 0$ otherwise. The second will be to show that $\chi(\widehat{HF}(Y, \mathfrak{s}))$ is independent of $\mathfrak{s}$.

For the first step, we begin by computing $\#(H_1(Y))$ as follows: choose a Morse function $f$ on $Y$ satisfying the conditions of Proposition 2.6. We get an induced handle decomposition of $Y$ and hence a cell decomposition of (a space equivalent to) $Y$. By Remark 2.4, the orientation on $Y$ determines one on $\Sigma$.

Consider the cellular chain complex associated to this cell decomposition. It has one generator each in degrees 0 and 3, and it has $g$ generators each in degrees 1 and 2. The boundary map from $C_1$ to $C_0$ is zero (since $H_0(Y) = \mathbb{Z}$), so we have $H_1(Y) = C_1 / \operatorname{im} C_2$. To pin down the boundary map from $C_2$ to $C_1$, we need to choose some orientations. Picking orientations for the $\alpha$ and $\beta$ curves will suffice, since this choice determines the signed intersection numbers between the attaching circles of the 2-handles and the belt circles of the 1-handles.

Label the generators of $C_1$ and $C_2$ as $x_1, \ldots, x_g$ and $y_1, \ldots, y_g$ respectively; the $x_i$ and $y_j$ correspond to the index 1 and index 2 critical points of $f$. We have the formula

$$\partial y_j = \sum_i \#(\alpha_i \cap \beta_j) x_i,$$

where the coefficients are the signed intersection numbers. In other words, with respect to our chosen bases, the map $C_2 \xrightarrow{\partial} C_1$ is a map $\mathbb{Z}^g \to \mathbb{Z}^g$ whose matrix $M$ has $ij^{th}$ entry $\#(\alpha_i \cap \beta_j)$. If $H_1(Y)$ is finite, then $M$ has nonzero determinant, and $\#(H_1(Y)) = \det M$. On the other hand, if $H_1(Y)$ is infinite, then $M$ must have determinant zero.

Hence, to complete the first step, we must show that $\chi(\widehat{HF}(Y)) = \det M$. Write $\det M$ as the expression $\sum_{\sigma \in S_g} \operatorname{sgn}(\sigma)\#(\alpha_1 \cap \beta_{\sigma(1)}) \cdots \#(\alpha_g \cap \beta_{\sigma(g)})$. The term $\operatorname{sgn}(\sigma)\#(\alpha_1 \cap \beta_{\sigma(1)}) \cdots \#(\alpha_g \cap \beta_{\sigma(g)})$ is a sum, over all intersection points $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ whose associated permutation is $\sigma$, of $\operatorname{sgn}(\sigma)\epsilon_1(\boldsymbol{x}) \cdots \epsilon_g(\boldsymbol{x})$, where the numbers $\epsilon_i(\boldsymbol{x}) = \pm 1$ were defined in Section 2.3.3. In other words, with respect to the chosen orientations on the $\alpha$ and $\beta$ curves (so that $\operatorname{gr}(\boldsymbol{x})$ is well-defined for $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$), we have $\det M = \sum_{\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \operatorname{gr}(\boldsymbol{x})$. We may write this sum as $\#\{\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} | \operatorname{gr}(\boldsymbol{x}) = 1\} - \#\{\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} | \operatorname{gr}(\boldsymbol{x}) = -1\} = \chi(\widehat{CF}(Y))$. But, by the usual argument, $\chi(\widehat{CF}(Y)) = \chi(\widehat{HF}(Y))$, so we have completed the first step.

For the second step, we want to show $\chi(\widehat{HF}(Y, \mathfrak{s}))$ is independent of $\mathfrak{s}$. Equivalently, for any $a \in H^2(Y)$, we want $\chi(\widehat{HF}(Y, \mathfrak{s})) = \chi(\widehat{HF}(Y, \mathfrak{s} + a))$. As in the statement of Proposition 2.19, though, for each $\alpha_i$ there is a dual curve $\alpha_i^*$ in $\Sigma$ which intersects $\alpha_i$ once and has zero intersection with the rest of the $\alpha$ curves. Also, all curves $\alpha_i$ and $\alpha_i^*$ have self-intersection zero. An elementary argument tells us that $\{[\alpha_1], \ldots, [\alpha_g], [\alpha_1^*], \ldots, [\alpha_g^*]\}$ form a basis for $H_1(\Sigma) \simeq \mathbb{Z}^{2g}$. Hence any element of $H_1(\Sigma)$ is in the span of the $[\alpha_i]$ and the $[\alpha_i^*]$, so any element of $H_1(\Sigma)$ is in the span of the $\alpha_i^*$ curves modulo the $\alpha$ curves. In particular, since $H_1(Y) = \frac{H_1(\Sigma)}{\langle [\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g] \rangle}$, we see that $\{[\alpha_1^*], \ldots, [\alpha_g^*]\}$ spans $H_1(Y)$. Hence $\{PD[\alpha_1^*], \ldots, PD[\alpha_g^*]\}$ spans $H^2(Y)$, and so we only need show that $\chi(\widehat{HF}(Y, \mathfrak{s})) = \chi(\widehat{HF}(Y, \mathfrak{s} + PD[\alpha_i^*]))$ for any $i$.

Choose a Heegaard diagram for $Y$ which is weakly admissible with basepoints $z$ and $z'$, where $z$ and $z'$ are two points on either side of $\alpha_i$, connected by a short arc $\delta$ which intersects $\alpha_i$ once and has zero intersection with the rest of the $\alpha$ curves. Such a Heegaard diagram always exists; see Section 5 of [6]. Then, by Proposition 2.19, the generators of $\widehat{CF}(Y, \mathfrak{s})$ with respect to the basepoint $z$ are the generators of $\widehat{CF}(Y, \mathfrak{s} + PD[\alpha_i^*])$ with respect to $z'$. While the maps in the complex may be different, the $\mathbb{Z}/2$ gradings of the generators are not, since they do not depend on basepoints. Since the Euler characteristic in homology may be computed on the chain level, we have $\chi(\widehat{HF}(Y, \mathfrak{s})) = \chi(\widehat{HF}(Y, \mathfrak{s} + PD[\alpha_i^*]))$ as claimed, where the left-hand side is computed using $\widehat{CF}(Y, \mathfrak{s})$ with the basepoint $z$ and the right-hand side is computed using $\widehat{CF}(Y, \mathfrak{s} + PD[\alpha_i])$ with the basepoint $z'$.

□

### 2.4.3 $\widehat{HF} = 0$ if and only if $HF^+ = 0$.

**Proposition 2.40.** *Suppose $Y$ is a three-manifold and $\mathfrak{s} \in \operatorname{Spin}^c(Y)$. Then $\widehat{HF}(Y, \mathfrak{s}) = 0$ if and only if $HF^+(Y, \mathfrak{s}) = 0$.*

*Proof.* The map $U$, viewed as an endomorphism of $CF^+(Y, \mathfrak{s})$, induces a short exact sequence of chain complexes

$$0 \to \widehat{CF}(Y, \mathfrak{s}) \to CF^+(Y, \mathfrak{s}) \xrightarrow{U} CF^+(Y, \mathfrak{s}) \to 0$$

and hence a corresponding long exact sequence on homology. If $HF^+(Y, \mathfrak{s}) = 0$, then clearly $\widehat{HF}(Y, \mathfrak{s}) = 0$. Conversely, suppose $\widehat{HF}(Y, \mathfrak{s}) = 0$. Then $HF^+(Y, \mathfrak{s}) \xrightarrow{U} HF^+(Y, \mathfrak{s})$ is an isomorphism. Suppose $[\boldsymbol{x}, i]$ is some element of $HF^+(Y, \mathfrak{s})$; then $U^{i+1}[\boldsymbol{x}, i] = 0$. But since $U$ is an isomorphism on homology, we must have $[\boldsymbol{x}, i] = 0$. This holds for arbitrary elements of $HF^+(Y, \mathfrak{s})$, so $HF^+(Y, \mathfrak{s}) = 0$. □

### 2.4.4 The adjunction inequality.

Finally, we state the adjunction inequality for $HF^+$ in the case of zero-surgeries on knots. It holds in greater generality, but this case is the only one we will need.

**Theorem 2.41.** *Let $K$ be a knot in $S^3$ with Seifert genus $g$. Then $HF^+(K_0, d) = 0$ for $d \geq g$.*

*Proof.* See Section 7 of [6]. □

## 2.5 Knot Floer homology.

Let $K$ be an oriented knot in $S^3$. We will define invariants of $K$, called the *knot Floer homology* groups of $K$, whose constructions are quite similar to those of the Heegaard Floer homology groups defined above.

### 2.5.1 Marked and doubly-pointed Heegaard diagrams.

Just as we needed to choose a pointed Heegaard diagram to define Heegaard Floer homology, we need to choose a marked Heegaard diagram for the knot $K$ in order to define knot Floer homology.

**Definition 2.42.** Let $K$ be a knot in $S^3$.

(a) A *marked Heegaard diagram* is a quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_g)$ are sets of attaching circles like usual and $m$ is a point in $\beta_g$ disjoint from the $\alpha$ curves.

(b) A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describes the knot $K$ if the following holds: the manifold resulting from attaching the $\alpha$ handlebody as usual and then attaching a 2-handle to each $\beta_i$, $1 \leq i \leq g-1$, is $S^3 \setminus \mathrm{nb}\, K$, and $\beta_g$ is a meridian $\mu$ for $K$ which intersects only one $\alpha$ curve (taken by convention to be $\alpha_g$). Another way of stating the first condition is that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}_0)$ describes $S^3 \setminus \mathrm{nb}\, K$, where $\boldsymbol{\beta}_0 = (\beta_1, \ldots, \beta_{g-1})$.

(c) A marked Heegaard diagram for $K$ is a marked diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describes $K$.

*Remark* 2.43. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describes $K$, then its associated 3-manifold must be $S^3$. Indeed, attaching a two-handle to the meridian of a knot and capping it off with a 3-ball always results in $S^3$.

One way to choose a marked Heegaard diagram for $K$ is to start with a bridge presentation of $K$:

**Definition 2.44.** A *bridge presentation* of a knot $K$, with $g$ bridges, is a projection of $K$ in which all crossings take place in $g$ designated "bridge segments" $a_1, \ldots, a_g \subset K$. At each crossing, the bridge segment $a_i$ is required to be the piece crossing under.

A basic result in knot theory ensures that if a knot $K$ has an embedding in $S^3$ which has $g$ maxima and $g$ minima in the $z$-direction, then $K$ admits a bridge presentation with $g$ bridges. Practically speaking, if $K$ is not too complicated, one can start with any projection of $K$, designate some bridge segments corresponding to maxima in a particular coordinate direction, and then "unwrap" the rest of the crossings. Figure 1 illustrates this procedure for the left-handed trefoil. In fact, the procedure for the $(2, 2k+1)$ torus knot works just as in this figure, and the resulting projection is called the Schubert normal form of the knot; see [12]. All our examples will be torus knots; the first two will be the trefoil and the $(2, 7)$ torus knot, and the Heegaard diagrams we will use for these come from Schubert normal forms.

Now we will describe how to obtain a marked Heegaard diagram from a bridge presentation. Start by viewing the plane of the projection, plus a point at $\infty$, as $S^2$. At each bridge segment $a_i$, attach a 1-handle to this $S^2$ at the two endpoints of $a_i$. Think of the 1-handle as going down below the plane. The result of attaching these $g$ handles is a genus $g$ surface; this will be our Heegaard surface $\Sigma$. To obtain the curve $\alpha_i$, close off $a_i$ with an arc inside the added handle. At this point, attaching 2-handles according to the $\alpha$ curves yields a genus $g$ handlebody. It may be visualized as the space below the plane of the projection, with $g$ tunnels removed.

Now, to obtain the $\beta$ curves, first note that $K \setminus \{a_1 \cup \cdots \cup a_g\}$ consists of $g$ components $b_1, \ldots, b_g$. Discard $b_g$. For $1 \leq i \leq g-1$, let $\beta_i$ be the boundary of a small tubular neighborhood of $b_i$ in $\Sigma$. Gluing in 2-handles according to these $\beta$ curves produces $S^3 \setminus \mathrm{nb}\, K$. Indeed, start at a point of $a_1$ and walk alternately "underground" through the tunnels according to the segments $a_i$ and "above ground" through the $\beta$-passageways according to the segments $b_i$. The resulting path traverses the whole of the knot except for the segment $b_g$. At the endpoints of $b_g$, we may imagine coming out from the tunnels and passageways. Taking a straight-line path between these two endpoints, above the rest of the crossings, amounts to traversing the remainder of $K$.

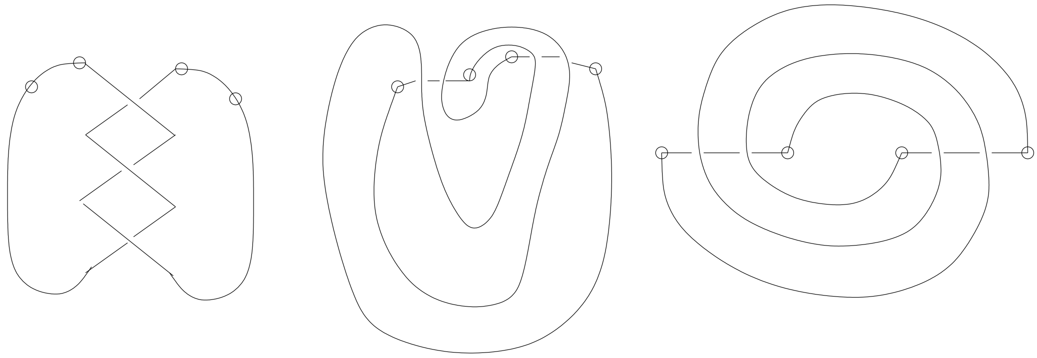

Figure 1: Producing a bridge presentation for the left-handed trefoil. The segments between the circles are the bridge segments.

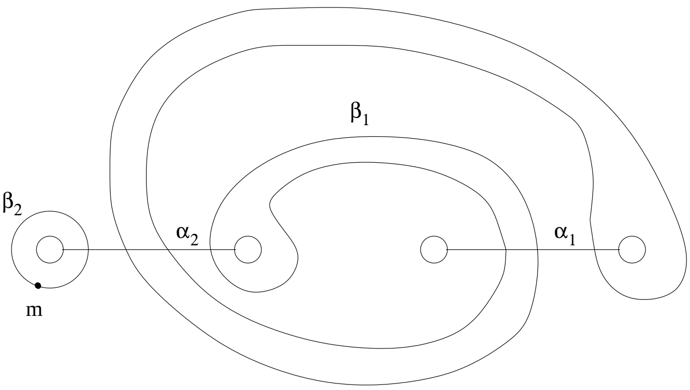


Figure 2: The resulting Heegaard diagram for the left-handed trefoil.

Finally, let $\beta_g$ be a small circle in $\Sigma$ around a mouth of the $\alpha_g$-tunnel. The curve $\beta_g$ is a meridian for $K$ intersecting only $\alpha_g$, and we obtain $S^3$ by performing the remaining gluings. Let $m$ be any point on $\beta_g$ disjoint from $\alpha_g$. We have informally proven the following proposition:

**Proposition 2.45.** *For any $K$, there exists a marked Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ for $K$.*

Figure 2 shows the result of this procedure applied to the left-handed trefoil.

Now, given a marked Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ for $K$, one can view all the possible longitudes for $K$ as curves in $\Sigma$, as follows. Push $m$ off $\beta_g$ in one direction to obtain a point $w$ and in the other direction to obtain a point $z$. Pick a path $\lambda'$ connecting $w$ and $z$ in the complement of the $\beta$ curves. We can close off $\lambda'$ with a short arc connecting $w$ and $z$ across $\beta_g$. The resulting closed curve $\lambda$ is untouched by the addition of handles according to the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_0$ curves; hence it is a curve in $\partial(S^3 \backslash \mathrm{nb}\, K) = \partial(\mathrm{nb}\, K)$. It intersects the meridian $\mu = \beta_g$ of $K$ once, so it is a longitude for $K$. As usual, the rest of the longitudes may be obtained from $\lambda$ by adding copies of $\mu$.

The above discussion also shows how a marked Heegaard diagram for $K$ gives rise to a "two-pointed" Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ for $S^3$, i.e. a Heegaard diagram for $S^3$ equipped with two basepoints $w$ and $z$ in the complement of the $\alpha$ and $\beta$ curves. The one subtlety is the ordering of $w$ and $z$. To fix which point is which, note that the (fixed) orientation on $K$ gives rise to an orientation on each longitude $\lambda$. Pick such a $\lambda$ as in the above paragraph, and relabel $w$ and $z$ if necessary so that the short arc crossing $\beta_g$ goes from $w$ to $z$. In this way, $w$ and $z$ are pinned down (up to isotopy in the complement of the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves).

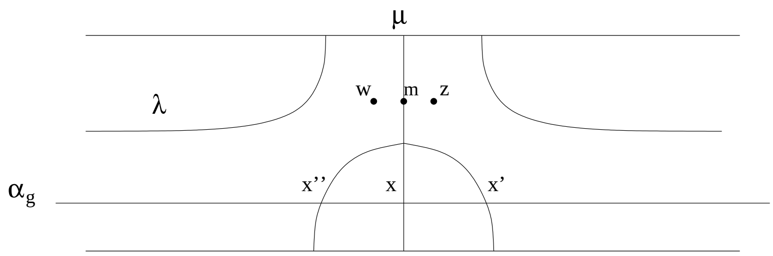


Figure 3: The region of intersection between $\lambda$ and $\mu$. The horizontal lines at top and bottom are glued together.

### 2.5.2 The knot Floer chain complexes.

Now we will define the knot Floer chain complex $CFK^\infty(S^3, K)$ as well as various subcomplexes and quotients.

**Definition 2.46.** Choose a marked Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ for $K$ with associated doubly-pointed diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$.

(a) The generators of $CFK^\infty(S^3, K)$ are elements $[\boldsymbol{x}, i, j]$, where $\boldsymbol{x}$ is an intersection point in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $i, j \in \mathbb{Z}$.

(b) The differential is given by:

$$\partial[\boldsymbol{x}, i, j] = \sum_{\{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, \phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) | \mu(\phi) = 1\}} \#\overline{\mathcal{M}}(\phi)[y, i - n_w(\phi), j - n_z(\phi)].$$

*Remark* 2.47. This definition makes sense for any doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, not just those coming from marked diagrams for knots. If we have an arbitrary doubly-pointed diagram, we will sometimes denote its chain complex as $CF^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$.

$CFK^\infty(S^3, K)$ trivially has two $\mathbb{Z}$-filtrations given by the $i$- and $j$-indices; these are true filtrations since if $\phi$ admits a holomorphic representative then $n_w(\phi)$ and $n_z(\phi)$ are nonnegative. Furthermore, a key observation is that it breaks into subcomplexes according to $\mathrm{Spin}^c$ structures on the zero-surgery $K_0$. We first need to discuss these.

Let $K_0$ be the zero-surgery of $K$, i.e. the manifold obtained from $S^3$ by surgery on $K$ with its Seifert framing. The Mayer-Vietoris sequence tells us that $H_1(K_0) = \mathbb{Z}$, so there are $\mathbb{Z}$ worth of $\mathrm{Spin}^c$ structures on $K_0$. Choose a Seifert surface $F$ for $K$, and let $\hat{F}$ be the closed surface in $K_0$ obtained by capping off $F$. The bijection $\mathrm{Spin}^c(K_0) \to \mathbb{Z}$ can be realized by sending $\mathfrak{t} \in \mathrm{Spin}^c(K_0)$ to $\frac{1}{2}\langle c_1(\mathfrak{t}), [\hat{F}]\rangle$. This bijection is independent of $F$; write $\mathfrak{t}_m$ for the $\mathrm{Spin}^c$ structure on $K_0$ such that $\frac{1}{2}\langle c_1(\mathfrak{t}_m), [\hat{F}]\rangle = m$. In particular, $\mathfrak{t}_0$ is the unique $\mathrm{Spin}^c$ structure with $c_1(\mathfrak{t}_0) = 0$.

Let $\lambda$ be the Seifert longitude for $K$, viewed as a curve in $\Sigma$ intersecting $\mu = \beta_g$ once and disjoint from all the other $\beta$ curves. Define $\gamma_g := \lambda$, and for $1 \leq i \leq g-1$, let $\gamma_i$ be a small isotopic translate of $\beta_i$ intersecting $\beta_i$ in two points with opposite sign. Then $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_g)$ is a set of attaching circles in $\Sigma$. Furthermore, the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ describes $K_0$, and we can choose $w$ as a basepoint ($z$ would work equally well, since there is now no $\beta_g$ blocking an isotopy of the two points). Like usual, we have a map $\mathfrak{s}_w : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}} \to \mathrm{Spin}^c(K_0)$. In this context, $\mathfrak{s}_w$ will always mean this map instead of the trivial map from $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ to $\mathrm{Spin}^c(S^3)$.

We can use the map $\mathfrak{s}_w : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}} \to \mathrm{Spin}^c(K_0)$ to define a map $\underline{\mathfrak{s}} : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(K_0)$. Strictly speaking, $\underline{\mathfrak{s}}$ depends on the marked point $m$ in the given marked Heegaard diagram for $K$, but since we will not need to vary this marked point, and since it will be useful to have $m$ available as an index, we suppress this dependence in our notation. After isotopy, we may assume that near the intersection of $\lambda = \gamma_g$ and $\mu = \beta_g$, $\lambda$ winds once around $\mu$, intersects, and then winds back. For an illustration, see Figure 3, which replicates Figure 3 of [5]. The point of intersection $x \in \beta_g \cap \alpha_g$ corresponds to two closest points $x'$ and $x'' \in \gamma_g \cap \alpha_g$. We may also assume that $\lambda$ does not contain the marked point $m$.

**Definition 2.48.** Let the triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be chosen as above.

(a) $\underline{\mathfrak{s}} : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(K_0)$ sends $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ to the Spin$^c$-structure $\mathfrak{s}_w(\boldsymbol{x}')$ on $K_0$ associated to $\boldsymbol{x}'$. Here $\boldsymbol{x}'$ denotes $\boldsymbol{x}$ with the $\beta_g$-component $x$ replaced by the point $x'$ defined above.

(b) Let $[\boldsymbol{x}, i, j] \in CFK^\infty(S^3, K)$. Then $\sigma[\boldsymbol{x}, i, j]$ is $\mathfrak{s}_m(\boldsymbol{x}) + (i - j)PD[\mu]$.

(c) For $\mathfrak{t} \in \mathrm{Spin}^c(K_0)$, $CFK^\infty(S^3, K, \mathfrak{t}) \subset CFK^\infty(S^3, K)$ is spanned by those generators $[\boldsymbol{x}, i, j]$ with $\sigma[\boldsymbol{x}, i, j] = \mathfrak{t}$.

*Remark* 2.49. In part (a) of Definition 2.48, note that $x''$ would work just as well. Indeed, let $\boldsymbol{x}''$ denote $\boldsymbol{x}$ with $x$ replaced by $x''$. There is an obvious domain representing a Whitney disk connecting $\boldsymbol{x}'$ to $\boldsymbol{x}''$ (or vice-versa), so $\epsilon(\boldsymbol{x}', \boldsymbol{x}'') = 0$ and $\mathfrak{s}_w(\boldsymbol{x}') = \mathfrak{s}_w(\boldsymbol{x}'')$. Thus, we do not need a way of distinguishing between $\boldsymbol{x}'$ and $\boldsymbol{x}''$. Also note that, by isotopy, we could use $\mathfrak{s}_z$ rather than $\mathfrak{s}_w$ to define $\underline{\mathfrak{s}}$.

Since Spin$^c$ structures on $K_0$ are rather abstract entities, it will be useful to have a reformulation of Spin$^c(K_0)$-differences between elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ in terms of something more concrete. To this end, the following lemma is very useful:

**Lemma 2.50.** *Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Let $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$. Then $\underline{\mathfrak{s}}(\boldsymbol{x}) - \underline{\mathfrak{s}}(\boldsymbol{y}) = (n_z(\phi) - n_w(\phi)) \cdot PD[\mu]$.*

*Proof.* See Lemma 2.5 of [5]. □

**Proposition 2.51.** *$CFK^\infty(S^3, K, \mathfrak{t})$ is a filtered subcomplex of $CFK^\infty(S^3, K)$, and hence*

$$CFK^\infty(S^3, K) = \oplus_{\mathfrak{t} \in \mathrm{Spin}^c(K_0)} CFK^\infty(S^3, K, \mathfrak{t})$$

*as a filtered complex.*

*Proof.* Immediate from Lemma 2.50 and the formula for the differential in $CFK^\infty(S^3, K, \mathfrak{t})$. □

*Remark* 2.52. We will rarely need to consider the full sum $\oplus_{\mathfrak{t} \in \mathrm{Spin}^c(K_0)} CFK^\infty(S^3, K, \mathfrak{t})$. Thus, at this point we will change notation. $CFK^\infty(S^3, K)$ will no longer refer to this sum; instead, it will be shorthand for $CFK^\infty(S^3, K, \mathfrak{t}_0)$. Also, if $m \in \mathbb{Z}$, we will often write $CFK^\infty(S^3, K, m)$ in place of $CFK^\infty(S^3, K, \mathfrak{t}_m)$.

*Remark* 2.53. Now that we are working with the fixed Spin$^c$ structure $\mathfrak{t}_0$, the equation $\sigma([\boldsymbol{x}, i, j]) = \underline{\mathfrak{s}}(\boldsymbol{x}) + (i - j)PD[\mu] = \mathfrak{t}_0$ uniquely determines $j$ once $\boldsymbol{x}$ and $i$ are known. Thus, the generators $[\boldsymbol{x}, i, j]$ of $CFK^\infty(S^3, K)$ correspond bijectively to generators $[\boldsymbol{x}, i]$ of $CF^\infty(S^3)$. In fact an examination of the differential shows $CFK^\infty(S^3, K)$ and $CF^\infty(S^3)$ are isomorphic as complexes. The effect of the extra index $j$ attached to generators of $CFK^\infty(S^3, K)$ is to give this complex another $\mathbb{Z}$-filtration. Also, since $CF^\infty(S^3)$ is absolutely $\mathbb{Z}$-graded, we get an absolute $\mathbb{Z}$-grading on $CFK^\infty(S^3, K)$. We will sometimes use the term "homological grading" as a synonym for this absolute grading.

From $CFK^\infty(S^3, K)$, which now means $CFK^\infty(S^3, K, \mathfrak{t}_0)$, we will define associated complexes $CFK^{\{i<0\}}(S^3, K)$, $CFK^{\{i\geq 0\}}(S^3, K)$, $CFK^{\{i=0\}}(S^3, K)$, and $\widehat{CFK}(S^3, K)$.

**Definition 2.54.** Let $K$ be as above.

(a) $CFK^{\{i<0\}}(S^3, K)$ is the subcomplex of $CFK^\infty(S^3, K)$ spanned by elements $[\boldsymbol{x}, i, j]$ with $i < 0$.

(b) $CFK^{\{i\geq 0\}}(S^3, K)$ is the quotient of $CFK^\infty(S^3, K)$ by $CFK^{\{i<0\}}(S^3, K)$.

(c) $CFK^{\{i=0\}}(S^3, K)$ is the subcomplex of $CFK^{\{i\geq 0\}}(S^3, K)$ spanned by elements with $i = 0$.

*Remark* 2.55. Just as in the $\infty$ case, $CFK^{\{i=0\}}(S^3, K)$ is isomorphic to $\widehat{CF}(S^3)$. As before, the extra index $j$ gives this complex a $\mathbb{Z}$-filtration, and the absolute $\mathbb{Z}$-grading on $\widehat{CF}(S^3)$ gives an absolute $\mathbb{Z}$-grading on $CFK^{\{i=0\}}(S^3, K)$.

**Definition 2.56.** $\widehat{CFK}(S^3, K)$ is the graded complex associated to $CFK^{\{i=0\}}(S^3, K)$ as a $\mathbb{Z}$-filtered complex. The piece at filtration level $d$ is denoted $\widehat{CFK}(S^3, K, d)$, with homology $\widehat{HFK}(S^3, K, d)$.

It might seem logical to use a name like $\widehat{CFK}^{\{j=d\}}(S^3, K)$ or even $CFK^{\{i=0,j=d\}}(S^3, K)$ for the piece of the complex at filtration level $d$. However, the convention is to use the name $\widehat{CFK}(S^3, K, d)$. This usage should not be confused with the earlier $CFK^\infty(S^3, K, m)$, where $m \in \mathbb{Z}$. Whereas generators $[\boldsymbol{x}, i, j]$ of this latter group satisfy $\sigma([\boldsymbol{x}, i, j]) = \mathfrak{t}_m$, generators $[\boldsymbol{x}, 0, d]$ of $\widehat{CFK}(S^3, K, d)$ always satisfy $\sigma([\boldsymbol{x}, 0, d]) = \mathfrak{t}_0$.

### 2.5.3 Conjugation symmetry.

$\widehat{HFK}$ is symmetric under conjugation of Spin$^c$ structures:

**Proposition 2.57.** *For $m \in \mathbb{Z}$ and $d \in Z$, we have the conjugation symmetry*

$$\widehat{HFK}_d(S^3, K, m) = \widehat{HFK}_{d-2m}(S^3, K, -m),$$

*where the subscripts denote absolute degrees.*

*Proof.* See Proposition 3.10 of [5]. $\square$

## 2.6 Holomorphic triangles and cobordisms.

Now that we have defined Heegaard Floer homology and knot Floer homology, we turn to maps between Heegaard Floer groups defined by counting holomorphic triangles in Heegaard triples. Such maps may be interpreted as maps induced from surgery cobordisms. The maps induced by cobordisms between rational homology three-spheres behave predictably with respect to the absolute $\mathbb{Q}$-grading.

### 2.6.1 Heegaard triples.

**Definition 2.58.** A *Heegaard triple* is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, where $\Sigma$ is an oriented genus-$g$ surface and $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ are three sets of attaching circles in $\Sigma$. A *pointed* Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is a Heegaard triple equipped with a basepoint $z$ disjoint from all attaching circles.

Suppose $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a Heegaard triple. There are three ways to choose two out of the three sets of circles $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ and glue handlebodies along them to produce a 3-manifold. We will call the resulting 3-manifolds $Y_{\boldsymbol{\alpha\beta}}$, $Y_{\boldsymbol{\beta\gamma}}$, and $Y_{\boldsymbol{\alpha\gamma}}$. In fact, whenever we have any Heegaard tuple, analogous notation will be used. Thus, for instance, if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ is a Heegaard quadruple, then there are six naturally associated 3-manifolds, and they will be denoted $Y_{\boldsymbol{\alpha\beta}}$, $Y_{\boldsymbol{\alpha\gamma}}$, $Y_{\boldsymbol{\alpha\delta}}$, $Y_{\boldsymbol{\beta\gamma}}$, $Y_{\boldsymbol{\beta\delta}}$, and $Y_{\boldsymbol{\gamma\delta}}$.

Starting with a pointed Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$, we want to produce a map $\widehat{HF}(Y_{\boldsymbol{\alpha\beta}}) \otimes \widehat{HF}(Y_{\boldsymbol{\beta\gamma}}) \to \widehat{HF}(Y_{\boldsymbol{\alpha\gamma}})$. The map will count holomorphic triangles in $\mathrm{Sym}^g \Sigma$. Inside $\mathrm{Sym}^g \Sigma$, we now have three tori, denoted $\mathbb{T}_{\boldsymbol{\alpha}}, \mathbb{T}_{\boldsymbol{\beta}}$, and $\mathbb{T}_{\boldsymbol{\gamma}}$.

**Definition 2.59.** Let $\Delta$ be the standard 2-simplex with vertices $v_\alpha, v_\beta$, and $v_\gamma$ (arranged clockwise) and edges $e_\alpha, e_\beta$, and $e_\gamma$ opposite them. Let $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\beta}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$, and $\boldsymbol{z} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$.

(a) A *Whitney triangle* connecting $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ is a map $\Delta \xrightarrow{\psi} \mathrm{Sym}^g \Sigma$ such that $\psi(v_\gamma) = \boldsymbol{x}$, $\psi(v_\alpha) = \boldsymbol{y}$, $\psi(v_\beta) = \boldsymbol{z}$, $\psi(e_\gamma) \subset \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $\psi(e_\alpha) \subset \mathbb{T}_{\boldsymbol{\beta}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$, and $\psi(e_\beta) \subset \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$.

(b) $\pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is the set of homotopy classes of Whitney triangles connecting $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$.

(c) If $\psi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, then $\mathcal{M}(\psi)$ is the moduli space of holomorphic representatives of $\psi$; its Maslov index is $\mu(\psi)$, and its signed intersection with $V_z$ is $n_z(\psi)$.

*Remark* 2.60. If $\psi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$, we can define the domain $D(\psi)$ of $\psi$ analogously to the domain of a Whitney disk. It is a formal linear combination of the components of $\Sigma \setminus (\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g \cup \gamma_1 \cup \cdots \cup \gamma_g)$.

We can now define the map associated to a Heegaard triple:

**Definition 2.61.** Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ be a pointed Heegaard triple. Define a map $F_{(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)}$ from $\widehat{CF}(Y_{\boldsymbol{\alpha\beta}}) \otimes_{\mathbb{Z}} \widehat{CF}(Y_{\boldsymbol{\beta\gamma}})$ to $\widehat{CF}(Y_{\boldsymbol{\alpha\gamma}})$ as follows: if $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\beta}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$, then

$$\hat{F}_{(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)}(\boldsymbol{x} \otimes \boldsymbol{y}) = \sum_{\{\boldsymbol{z} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}, \psi \in \pi_2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) | \mu(\psi)=0, n_z(\psi)=0\}} \#(\mathcal{M}(\psi)) \cdot z.$$

It is a nontrivial result that $F_{(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)}$ is a chain map, with respect to the usual differential on the tensor product of complexes, and hence induces a map on homology. There is an analogous map from $CF^+(Y_{\boldsymbol{\alpha\beta}}) \otimes_{\mathbb{Z}[U]} CF^+(Y_{\boldsymbol{\beta\gamma}})$ to $CF^+(Y_{\boldsymbol{\alpha\gamma}})$ defined by

$$F^+_{(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)}([\boldsymbol{x}, i] \otimes [\boldsymbol{y}, j]) = \sum_{\{\boldsymbol{z} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}, \psi \in \pi_2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) | \mu(\psi)=0\}} \#(\mathcal{M}(\psi)) \cdot [z, i + j - n_z(\psi)].$$

This map also induces a map on homology.

### 2.6.2 Surgery cobordisms.

We are interested primarily in surgeries on knots in $S^3$. However, when discussing the surgery exact triangle in Section 4, we will want to consider further surgeries on knots in these $S^3$-surgeries. Thus, in this section, we will let $K$ be a null-homologous knot in an arbitrary (closed, oriented) 3-manifold $Y$.

Suppose $\lambda$ is a longitude for $K$ having intersection number $\pm 1$ with $[\mu]$ inside $\partial(\mathrm{nb}\, K)$; represent $\lambda$ by an embedded circle in $\partial(\mathrm{nb}\, K)$. One can define a cobordism $W$ from $Y$ to the surgered manifold $K_\lambda$ by starting with $Y \times [0, 1]$ and attaching a two-handle $H = D^2 \times D^2$ to $Y \times \{1\}$. The two-handle is attached as follows: the attaching circle $S^1 \times \{0\}$ is glued to $K$, and the rest of $S^1 \times D^2$ fills out $\mathrm{nb}\, K$ in such a way that $S^1 \times \{1\}$ is glued to $\lambda$. The operation of gluing the 2-handle replaces the boundary component $Y \times \{1\}$ with $K_\lambda$: points in the interior of $\mathrm{nb}\, K$ do not contribute to the boundary of $W$, and "new" boundary points come from the $D^2 \times S^1$ component of $\partial H$. Because of how we glued $H$, the resulting boundary component of $W$ is exactly $K_\lambda$.

Orient $W$ so that it is a cobordism from $Y$ to $K_\lambda$. In general, any cobordism between 3-manifolds $Y_1$ and $Y_2$ should induce a map on Heegaard Floer homology. We will not define this induced map for all cobordisms; we will only define it for knot surgery cobordisms such as $W$. To do so, we will construct a Heegaard triple from the surgery data, and then we will make use of the constructions from Section 2.6.1.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram describing $K$ as in Section 2.5.1. The longitude $\lambda$ may be realized as a closed curve in $\Sigma$; call it $\gamma_g$. For $1 \le i \le g - 1$, let $\gamma_i$ be an isotopic translate of $\beta_i$, intersecting $\beta_i$ in two points with opposite signs. Then $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_g)$ is a set of attaching circles in $\Sigma$. Our Heegaard triple is defined to be $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$.

Consider the manifolds $Y_{\boldsymbol{\alpha\beta}}$, $Y_{\boldsymbol{\beta\gamma}}$, and $Y_{\boldsymbol{\alpha\gamma}}$ associated to this triple. Clearly $Y_{\boldsymbol{\alpha\beta}}$ is just $S^3$. Also, $Y_{\boldsymbol{\alpha\gamma}}$ is the surgered manifold $K_\lambda$. Indeed, the difference between $Y_{\boldsymbol{\alpha\beta}}$ and $Y_{\boldsymbol{\alpha\gamma}}$ is that we removed a 3-ball and the 2-handle corresponding to $\beta_g$ from $Y_{\boldsymbol{\alpha\beta}}$, amounting to the removal of $\mathrm{nb}\, K$ from $S^3$, and glued back a solid torus in such a way that $\gamma_g = \lambda$ bounds a disk. Finally, $Y_{\boldsymbol{\beta\gamma}}$ is just $\#^{g-1}(S^2 \times S^1)$. This final identification holds because, first of all, $\beta_i$ only intersects $\gamma_i$ and no other $\gamma$ curve, so we can separate the situation into $g$ connected summands. Second of all, in the summands for $1 \le i \le g - 1$, the intersection pattern between $\beta_i$ and $\gamma_i$ looks just like the standard admissible Heegaard diagram for $S^2 \times S^1$, while in the $g^{th}$ summand it looks like the standard diagram for $S^3$.

For $1 \le i \le g - 1$, label the two points of $\beta_i \cap \gamma_i$ as $y_i^\pm$. More precisely, label them so that the two obvious ways of travelling from $y_i^-$ to $y_i^+$ along $\beta_i$ and then back to $y_i^-$ along $\gamma_i$ each produce a circle which bounds an oriented disk agreeing with the orientation of $\Sigma$. This choice ensures that in $\widehat{CF}$ of the $S^2 \times S^1$ summand corresponding to $\beta_i$ and $\gamma_i$, both differentials map $y_i^-$ to $y_i^+$, so we have $\partial y_i^+ = 0$.

Let $y_g$ be the unique intersection point between $\beta_g$ and $\gamma_g$. Define $\Theta_{\boldsymbol{\beta\gamma}}$ to be $\{y_1^+, \ldots, y_{g-1}^+, y_g\}$, an element of $\mathbb{T}_{\boldsymbol{\beta}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$. We claim that $\partial \Theta_{\boldsymbol{\beta\gamma}} = 0$ in $\widehat{CF}(\#^{g-1}(S^2 \times S^1))$. This fact is a consequence of the following proposition from [6]:

**Proposition 2.62.** *If* $Y = Y_1 \# Y_2$*, then* $\widehat{CF}(Y) = \widehat{CF}(Y_1) \otimes_{\mathbb{Z}} \widehat{CF}(Y_2)$.

*Proof.* See Proposition 6.1 of [6]. □

Using induction, we see that we can write $\Theta_{\boldsymbol{\beta\gamma}}$ as $y_1^+ \otimes \cdots \otimes y_{g-1}^+ \otimes y_g$ under the appropriate identification, so $\partial\Theta_{\boldsymbol{\beta\gamma}} = [(\partial y_1^+)\otimes\cdots\otimes y_{g-1}^+\otimes y_g]+\ldots+[y_1^+\otimes\cdots\otimes(\partial y_{g-1}^+)\otimes y_g]+[y_1^+\otimes\cdots\otimes y_{g-1}^+\otimes(\partial y_g)]$. But $\partial y_g = 0$ and $\partial y_i^+ = 0$ for $1 \leq i \leq g-1$, so $\partial\Theta_{\boldsymbol{\beta\gamma}} = 0$. Hence $\Theta_{\boldsymbol{\beta\gamma}}$ represents a homology class in $\widehat{HF}(Y_{\boldsymbol{\beta\gamma}})$.

There is an obvious analogue of the above computation for $HF^+$: since the differential of $[\Theta_{\boldsymbol{\beta\gamma}}, 0]$ in $CF^+(Y_{\boldsymbol{\beta\gamma}})$ counts the same disks as the differential of $\Theta_{\boldsymbol{\beta\gamma}}$ in $\widehat{CF}(Y_{\boldsymbol{\beta\gamma}})$, it is also zero. Hence $[\Theta_{\boldsymbol{\beta\gamma}}, 0]$ represents a homology class in $HF^+(Y_{\boldsymbol{\beta\gamma}})$.

We can plug these distinguished elements of $\widehat{HF}(Y_{\boldsymbol{\beta\gamma}})$ and $HF^+(Y_{\boldsymbol{\beta\gamma}})$ into the maps defined in Section 2.6.1 to define the maps induced by the surgery cobordisms:

**Definition 2.63.** Let $W$ be the surgery cobordism from $Y$ to $K_\lambda$.

(a) The map $\Phi_W : \widehat{HF}(Y) \to \widehat{HF}(K_\lambda)$ sends $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ to $\hat{F}_{(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)}(\boldsymbol{x} \otimes \Theta_{\boldsymbol{\beta\gamma}})$.

(b) The map $\Phi_W : HF^+(Y) \to HF^+(K_\lambda)$ sends $[\boldsymbol{x}, i] \in (\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}})\times\mathbb{Z}^{\geq 0}$ to $F^+_{(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)}([\boldsymbol{x}, i]\otimes [\Theta_{\boldsymbol{\beta\gamma}}, 0])$.

Note that to avoid a cluttering of notation, we will use $\Phi_W$ to denote both the map on $\widehat{HF}$ and the map on $HF^+$. There are analogous maps on $HF^-$ and $HF^\infty$.

### 2.6.3 Spin$^c$ structures on cobordisms.

Here we cite without proof the results needed concerning Spin$^c$ structures on cobordisms. Let $W : Y_1 \to Y_2$ be a cobordism; there is an induced map $\Phi_W$ on Heegaard Floer homology. We have only defined $\Phi_W$ when $W$ is a knot surgery cobordism, and we will only need this case, but the first proposition in this section holds for any cobordism.

The key fact is that $\Phi_W$ splits up as a sum of maps $\Phi_{W,\mathfrak{r}}$ according to Spin$^c$ structures $\mathfrak{r}$ on $W$. Unfortunately, the interpretation of Spin$^c$ structures in terms of nowhere-vanishing vector fields only works for 3-manifolds. Rather than defining Spin$^c$ structures on 4-manifolds, we simply state the relevant proposition without proof, in the case where $Y_1$ and $Y_2$ are rational homology three-spheres.

**Proposition 2.64.** *Suppose $W$ is a cobordism from $Y_1$ to $Y_2$, where $Y_1$ and $Y_2$ are rational homology three-spheres.*

*(a) There is a map $c_1 : \mathrm{Spin}^c(W) \to H^2(W)$ called the first Chern class. It gives a bijective correspondence between* Spin$^c$ *structures on $W$ and $\{\alpha \in H^2(W) | \alpha(x) = x \cdot x \mod 2\ \forall x \in H_2(W)\}$, where $\cdot$ denotes the intersection pairing on $H_2(W)$.*

*(b) There is a restriction map which associates to $\mathfrak{r} \in \mathrm{Spin}^c(W)$ a* Spin$^c$ *structure $\mathfrak{r}|_{Y_1}$ on $Y_1$ and a* Spin$^c$ *structure $\mathfrak{r}|_{Y_2}$ on $Y_2$.*

*(c) The map $\Phi_W : HF^\circ(Y_1) \to HF^\circ(Y_2)$ is a sum, over all $\mathfrak{r} \in \mathrm{Spin}^c(W)$, of maps $\Phi_{W,\mathfrak{r}} : HF^\circ(Y_1, \mathfrak{s}_1) \to HF^\circ(Y_2, \mathfrak{s}_2)$, where $\mathfrak{s}_i = \mathfrak{r}|_{Y_i}$ and $HF^\circ$ stands for $\widehat{HF}$ or $HF^+$.*

*(d) For $\mathfrak{r} \in \mathrm{Spin}^c(W)$, the map $\Phi_{W,\mathfrak{r}}$ shifts the absolute $\mathbb{Q}$-grading by $\frac{c_1(\mathfrak{r})^2-2\chi(W)-3\sigma(W)}{4}$, where $\chi(W)$ is the Euler characteristic of $W$ and $\sigma(W)$ is the signature of the intersection pairing on $H_2(W)$.*

*Remark* 2.65. The intersection pairing on $H_2(W)$ is a well-defined integer-valued pairing. If $x, y \in H_2(W)$, then $PD[x]$ and $PD[y]$ are in $H^2(W, \partial W)$. Their cup product $PD[x] \cup PD[y]$ is in $H^4(W, \partial W)$, so it can be paired with the fundamental class $[W] \in H_4(W, \partial W)$ to produce an integer.

The cup product pairing on $H^2(W)$, on the other hand, takes some care to define. Consider the inclusion of pairs $j : (W, \emptyset) \to (W, \partial W)$. Consideration of the exact cohomology sequence of $(W, \partial W)$ with $\mathbb{Q}$ coefficients shows that $H^2(W, \partial W; \mathbb{Q}) \xrightarrow{j^*} H^2(W; \mathbb{Q})$ is an isomorphism. Now, if $a \in H^2(W; \mathbb{Q})$, we have $(j^*)^{-1}(a) \in H^2(W, \partial W; \mathbb{Q})$. For $a$ and $b$ in $H^2(W; \mathbb{Q})$, we may define their cup product pairing to be $((j^*)^{-1}(a) \cup (j^*)^{-1}(b))[W]$. The result is a rational number, not an integer. In particular, $c_1(\mathfrak{r})^2$ in the above proposition may not be an integer.

**Theorem 2.66.** *Let $W : Y_1 \to Y_2$ be a cobordism and let $\mathfrak{r} \in \mathrm{Spin}^c(W)$. There is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & HF^-(Y_1, \mathfrak{r}|_{Y_1}) \arrow[r] \arrow[d, "\Phi_{W,\mathfrak{r}}"] & HF^\infty(Y_1, \mathfrak{r}|_{Y_1}) \arrow[r] \arrow[d, "\Phi_{W,\mathfrak{r}}"] & HF^+(Y_1, \mathfrak{r}|_{Y_1}) \arrow[r] \arrow[d, "\Phi_{W,\mathfrak{r}}"] & 0 \\
0 \arrow[r] & HF^-(Y_2, \mathfrak{r}|_{Y_2}) \arrow[r] & HF^\infty(Y_2, \mathfrak{r}|_{Y_2}) \arrow[r] & HF^+(Y_2, \mathfrak{r}|_{Y_2}) \arrow[r] & 0
\end{tikzcd}$$

*where the horizontal sequences come from Theorem 2.35.*

For surgery cobordisms, we can describe the splitting $\Phi_W = \sum_{\mathfrak{r}} \Phi_{W,\mathfrak{r}}$ in terms of holomorphic triangles.

**Proposition 2.67.** *Suppose $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is the Heegaard triple associated to $\lambda$-framed surgery on a knot $K$ in $Y$, and let $W$ be the surgery cobordism from $Y$ to $K_\lambda$.*

*(a) For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$, $\pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ is nonempty if and only if there exists $\mathfrak{r} \in \mathrm{Spin}^c(W)$ such that $\mathfrak{r}|_Y = \mathfrak{s}_z(\boldsymbol{x})$ and $\mathfrak{r}|_{K_\lambda} = \mathfrak{s}_z(\boldsymbol{y})$. In such a situation, we will say $\mathfrak{s}_z(\boldsymbol{x})$ is* cobordant *to $\mathfrak{s}_z(\boldsymbol{y})$.*

*(b) If $\pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ is nonempty, there is a natural mapping $\pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y}) \to \mathrm{Spin}^c(W)$, denoted $\psi \mapsto \mathfrak{s}_z(\psi)$, such that for all $\psi$ we have $\mathfrak{s}_z(\psi)|_Y = \mathfrak{s}_z(\boldsymbol{x})$ and $\mathfrak{s}_z(\psi)|_{K_\lambda} = \mathfrak{s}_z(\boldsymbol{y})$.*

*(c)*
$$\Phi_{W,\mathfrak{r}}(\boldsymbol{x}) = \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}, \psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y}) | \mu(\psi)=0, n_z(\psi)=0, \mathfrak{s}_z(\psi)=\mathfrak{r}\}} \#(\mathcal{M}(\psi)) \cdot \boldsymbol{y}.$$

### 2.6.4 Cobordisms from $S^3$ to $K_p$.

When $K$ is a knot in $S^3$ and $\lambda = \lambda_{Seif} + p\mu$, where $p \neq 0$, the surgery cobordism $W_p$ satisfies $H_2(W_p) = \mathbb{Z}$. To find a generator for this group, choose a Seifert surface $F$ for $K$. View it as living inside $S^3 \times I$ such that $\partial F \subset S^3 \times \{1\}$. Cap it off inside the added 2-handle $D^2 \times D^2$ to obtain a closed surface $S$ in $W_p$. The homology class $[S]$ generates $H_2(W_p)$, and it satisfies $[S] \cdot [S] = -|p|$. Let $[S]^* \in H^2(W_p) \simeq \mathbb{Z}$ denote the element dual to $[S]$, i.e. the element such that $[S]^*([S]) = +1$. Then $[S]^*$ generates $H^2(W_p)$. Furthermore, by Proposition 2.64, $\mathrm{Spin}^c$ structures on $W_p$ correspond to those elements $k[S]^*$ of $H^2(W)$ satisfying $k[S]^*(x) = x \cdot x \mod 2$ for all $x \in H_2(W)$. But this equation amounts to requiring that $k = k[S]^*([S]) = [S] \cdot [S] = -|p| \mod 2$. Hence, if $p$ is even, then $\mathrm{Spin}^c$ structures on $W_p$ correspond to $\{(2k)[S]^* | k \in \mathbb{Z}\}$, and if $p$ is odd, $\mathrm{Spin}^c$ structures on $W_p$ correspond to $\{(2k+1)[S]^* | k \in \mathbb{Z}\}$. Either way, if $\mathfrak{r}$ is a $\mathrm{Spin}^c$ structure on $W_p$ with $c_1(\mathfrak{r}) = k[S]^*$, then the quantity $\langle c_1(\mathfrak{r}), [S] \rangle - p = k - p$ is even. For $m \in \mathbb{Z}$, define $\mathfrak{r}_m$ to be the $\mathrm{Spin}^c$ structure on $W_p$ such that $\langle c_1(\mathfrak{r}_m), [S] \rangle - p = 2m$.

**Proposition 2.68.** *For $m, m' \in \mathbb{Z}$, we have $\mathfrak{r}_m|_{K_p} = \mathfrak{r}_{m'}|_{K_p}$ if and only if $m \equiv m' \mod |p|$.*

We already know (from the Mayer-Vietoris sequence) that there are $|p|$ distinct $\mathrm{Spin}^c$ structures on $K_p$, and now Proposition 2.68 gives us a way to label them. Namely, for $[m] \in \mathbb{Z}/p$, let $\mathfrak{s}_{[m]} := \mathfrak{r}_m|_{K_p}$. We will often refer to $\mathfrak{s}_{[m]}$ simply as $[m]$; thus, we have the formula $\widehat{HF}(K_p) = \sum_{[m] \in \mathbb{Z}/p} \widehat{HF}(K_p, [m])$. In particular, note that all $\mathrm{Spin}^c$ structures on $K_p$ are cobordant to the unique $\mathrm{Spin}^c$ structure on $S^3$.

Finally, as an addendum to Remark 2.65, note that $H_2(W_p) \xrightarrow{j_*} H_2(W_p, \partial W_p)$ is multiplication by $\pm p$. Correspondingly, $H^2(W_p, \partial W_p) \xrightarrow{j^*} H^2(W_p)$ is also multiplication by $\pm p$. From this formula, we can see explicitly that $j^*$ is an isomorphism with $\mathbb{Q}$ coefficients. We can also use it to compute $(c_1(\mathfrak{r}_m))^2$, which will be useful in future arguments:

**Proposition 2.69.** *Let $\mathfrak{r}_m$ be the $\mathrm{Spin}^c$ structure on $W_p$ defined above. Then*

$$(c_1(\mathfrak{r}_m))^2 = -\frac{1}{p}(2m+p)^2.$$

*Proof.* We first need to know what $([S]^*)^2$ is. By Remark 2.65, we look at $(j^*)^{-1}([S]^*)$. This element is some rational multiple of the generator $PD[S]$ of $H^2(W,\partial W;\mathbb{Q})$. Above, we said that $j^*$ was multiplication by $p$. In other words, it sends the generator $PD[S]$ of $H^2(W,\partial W)$ to $\pm p$ times the generator $[S]^*$ of $H^2(W)$. Thus, $(j^*)^{-1}([S]^*) = \pm\frac{1}{p}PD[S]$. We can now compute:

$$
\begin{aligned}
([S]^*)^2 &= \frac{1}{p^2}(PD[S]\cup PD[S])[W] \\
&= \frac{1}{p^2}[S]\cdot[S] \\
&= -\frac{1}{p}.
\end{aligned}
$$

Thus, $c_1(\mathfrak{s}_w(\psi))^2 = ((2m+p)[S]^*)^2 = -\frac{1}{p}(2m+p)^2$. $\square$

#### 2.6.5 Cobordisms from $K_p$ to $S^3$.

Inside $K_p = (S^3 \setminus \mathrm{nb}\,K)\cup_{\partial(\mathrm{nb}\,K)} (D^2\times S^1)$, there is a knot $K'$ given by $\{0\}\times S^1$. If $\mu$ denotes the meridian of $K$, then $\mu$ is a longitude for $K'$. Performing $\mu$-framed surgery on $K'$ results in $S^3$. We thus have a surgery cobordism from $K_p$ to $S^3$. This cobordism will come up several times in the rest of the paper; for notational convenience, we will just call it $W$ and suppress the dependence on $p$.

On the other hand, by basic Morse theory, one can see that $W$ is actually the same four-manifold as $W_p$. Thus we may apply the results of Section 2.6.5 to $W$. As before, $[S]$ is a generator of $H_2(W)$, and $[S]\cdot[S] = -p$. The Spin$^c$ structures on $W$ are the $\mathfrak{r}_m$ defined in Section 2.6.4. They satisfy $\langle c_1(\mathfrak{r}_m),[S]\rangle - p = 2m$, and we have $\mathfrak{r}_m|_{K_p} = [m]$.

#### 2.6.6 Area filtrations.

Because Heegaard triples play an important role in defining area filtrations on Heegaard Floer homology groups, and because area filtrations will be relevant in both Section 3 and Section 4, we discuss these filtrations here.

The most basic type of area filtration comes from areas of disks in an ordinary Heegaard diagram. Let $Y$ be a 3-manifold, and let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},z)$ be a Heegaard diagram for $Y$. For the sake of simplicity, assume $Y$ is an integer homology 3-sphere (in fact, for our applications, we will only need to consider $Y = S^3$). Suppose we have an area form on $\Sigma$ such that all periodic domains have zero area. Choose $\boldsymbol{x}_0 \in \mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$. We can define a filtration on $\widehat{CF}(Y)$ as follows: for $\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$, choose $\phi\in\pi_2(\boldsymbol{x}_0,\boldsymbol{x})$ with $n_z(\phi)=0$ (such $\phi$ exists since there is a unique Spin$^c$ structure on $Y$, so all $\epsilon$-differences are 0). Define

$$\mathcal{F}(\boldsymbol{x}) = -\mathcal{A}(D(\phi)),$$

where $\mathcal{A}$ denotes the signed area of a domain in $\Sigma$. The function $\mathcal{F}$ is well-defined since if $\phi'$ is another element of $\pi_2(\boldsymbol{x}_0,\boldsymbol{x})$ with $n_z(\phi)=0$, then the domains of $\phi$ and $\phi'$ differ by a periodic domain $P$, and $\mathcal{A}(P)=0$ by assumption.

For $\mathcal{F}$ to be a filtration on the chain complex $\widehat{CF}(Y)$, we want to ensure the differential does not increase $\mathcal{F}$. Suppose $\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$ and $\phi\in\pi_2(\boldsymbol{x}_0,\boldsymbol{x})$ with $n_z(\phi)=0$. If $\boldsymbol{y}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$, and $\phi'\in\pi_2(\boldsymbol{x},\boldsymbol{y})$ represents a differential from $\boldsymbol{x}$ to $\boldsymbol{y}$, then $\phi*\phi'\in\pi_2(\boldsymbol{x}_0,\boldsymbol{y})$, and $n_z(\phi*\phi')=0$, so we can use the domain of $\phi*\phi'$ to compute $\mathcal{F}(\boldsymbol{y})$. But $D(\phi*\phi') = D(\phi)+D(\phi')$, so $-\mathcal{A}(D(\phi*\phi')) = -\mathcal{A}(D(\phi)) - \mathcal{A}(D(\phi')) = \mathcal{F}(\boldsymbol{x}) - \mathcal{A}(D(\phi')) < \mathcal{F}(\boldsymbol{x})$ since domains of disks representing differentials have all nonnegative coefficients. Thus, the differential decreases $\mathcal{F}$.

Now suppose we have a Heegaard triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},z)$, where $\Sigma$ is again equipped with an area form such that all periodic domains have zero area. We require that either $Y_{\boldsymbol{\alpha\beta}}$ or $Y_{\boldsymbol{\alpha\gamma}}$ is a homology 3-sphere; it follows that the $\alpha$, $\beta$, and $\gamma$ curves span $H_1(\Sigma)$. We will also restrict attention to the case when $Y_{\boldsymbol{\beta\gamma}} = \#^{g-1}(S^2\times S^1)$, which is always true when the Heegaard triple arises from a knot surgery. As above, we have an element $\Theta_{\boldsymbol{\beta\gamma}}$ in $\widehat{CF}(Y_{\boldsymbol{\beta\gamma}})$ which represents an element of homology.

Given this data, a construction analogous to the one above, using holomorphic triangles rather than holomorphic disks, defines an area filtration on $Y_{\boldsymbol{\alpha\gamma}}$. Choose $\boldsymbol{x}_0\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$. For $\boldsymbol{y}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\gamma}}$, pick

$\psi \in \pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ with $n_z(\psi) = 0$. Such $\psi$ exists since the $\alpha, \beta$, and $\gamma$ curves span $H_1(\Sigma)$. Indeed, we may construct the domain of $\psi$ as in Section 2.2.5. We may now define

$$\mathcal{F}(\boldsymbol{y}) = -\mathcal{A}(D(\psi)).$$

Again, $\mathcal{F}$ is well-defined since all periodic domains have zero area. To see that the differential on $\widehat{CF}(Y_{\boldsymbol{\alpha\gamma}})$ decreases $\mathcal{F}$, let $\boldsymbol{y}, \boldsymbol{y'} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$ and suppose $\phi \in \pi_2(\boldsymbol{y}, \boldsymbol{y'})$ represents a differential. Choose $\psi \in \pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ with $n_z(\psi) = 0$. Then $\psi * \phi$ is an element of $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y'})$ satisfying $n_z(\psi * \phi) = 0$, so $\psi * \phi$ may be used to compute $\mathcal{F}(\boldsymbol{y'})$. But then $\mathcal{F}(\boldsymbol{y'}) = \mathcal{F}(\boldsymbol{y}) - \mathcal{A}(D(\phi)) < \mathcal{F}(\boldsymbol{y})$.

Finally, in the proof of the exact triangle in Section 4, we will consider a third filtration using squares. This time, we start with a Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ with an area form on $\Sigma$. We require that all periodic domains have zero area and that one of the three manifolds $Y_{\boldsymbol{\alpha\beta}}$, $Y_{\boldsymbol{\alpha\gamma}}$, or $Y_{\boldsymbol{\alpha\delta}}$ is an integer homology 3-sphere. We will also assume that both $Y_{\boldsymbol{\beta\gamma}}$ and $Y_{\boldsymbol{\gamma\delta}}$ are $\#^{g-1}(S^2 \times S^1)$; we have elements $\Theta_{\boldsymbol{\beta\gamma}} \in \widehat{CF}(Y_{\boldsymbol{\beta\gamma}})$ and $\Theta_{\boldsymbol{\gamma\delta}} \in \widehat{CF}(Y_{\boldsymbol{\gamma\delta}})$ which represent elements of homology. Choose $\boldsymbol{x}_0 \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. For $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}}$, pick $\varphi \in \pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}}, \Theta_{\boldsymbol{\gamma},\boldsymbol{\delta}}, \boldsymbol{y})$, the set of homotopy classes of maps of squares (defined analogously to disks and triangles), with $n_z(\varphi) = 0$. Such $\varphi$ exists as long as $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}}, \Theta_{\boldsymbol{\gamma},\boldsymbol{\delta}}, \boldsymbol{y})$ is nonempty, which is true since the $\alpha$, $\beta$, $\gamma$, and $\delta$ curves span $H_1(\Sigma)$. Define

$$\mathcal{F}(\boldsymbol{y}) = -\mathcal{A}(D(\varphi)).$$

The same arguments as before show that $\mathcal{F}$ is a well-defined filtration on $\widehat{CF}(Y_{\boldsymbol{\alpha\delta}})$.

# 3 Computing the Heegaard Floer homology of large integer surgeries on knots in $S^3$.

## 3.1 Introduction

The purpose of this section will be to answer the following question: given a knot $K$ in $S^3$, how does one compute $HF^+(K_p)$ when $p$ is large in absolute value? We will give an answer in terms of the knot Floer chain complexes of $K$. In the spirit of Section 2.5.2, we make the following definition:

**Definition 3.1.** Let $K$ be a knot in $S^3$ and let $n$ be an integer. The complex $CFK^{\{i\geq 0, j \geq n\}}(S^3, K)$ is the quotient of $CFK^\infty(S^3, K)$ by the subcomplex generated by elements $[\boldsymbol{x}, i, j]$ with $i < 0$ and $j < n$.

*Remark* 3.2. From now on, we will not explicitly spell out the definitions of such quotients or subcomplexes of $CFK^\infty(S^3, K)$. They are all analogous to this one.

**Theorem 3.3.** *Let $K$ be a knot in $S^3$. Let $g$ be its genus. There exists an integer $N$ such that for all $p \geq N$, the following equation holds:*

$$HF^+_l(K_{-p}, [m]) \simeq \begin{cases} H_{l - \frac{-(2m+p)^2+p}{4p}}(CFK^{\{i\geq 0, j\geq -m\}}(S^3, K)) & |m| \leq g, \\ HF^+_{l - \frac{-(2m+p)^2+p}{4p}}(S^3) & \textit{otherwise,} \end{cases}$$

*for $[m] \in \mathbb{Z}/p$. The subscripts denote absolute $\mathbb{Q}$-gradings.*

Theorem 3.3 is a relatively straightforward consequence of Theorem 3.4 below, which identifies all variants of $HF^\circ(K_{-p}, [m])$ but does not explicitly mention absolute gradings. Theorem 3.4 uses a map $\Phi : CFK^\infty(S^3, K) \to CF^\infty(K_{-p})$, which we now define. Pick a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ associated to the surgery data $(K, \lambda_p)$ as in Section 2.6.2. More precisely, choose $\lambda_p$ to follow $\lambda_{Seif}$ closely except near $\mu = \beta_g$ and then to wind tightly $p$ times around $\mu$, intersecting somewhere in the middle (after about $p/2$ winds). Mark a point $m \in \beta_g$; then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ is a marked Heegaard diagram for $K$, and there are associated basepoints $w$ and $z$. For a generator $[\boldsymbol{x}, i, j]$ of $CFK^\infty(S^3, K)$, define

$$\Phi([\boldsymbol{x}, i, j]) = \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}, \psi \in \pi_2 \boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y} | \mu(\psi) = 0, n_w(\psi) - n_z(\psi) = i - j\}} \#\mathcal{M}(\psi)[\boldsymbol{x}, i - n_w(\psi)].$$

**Theorem 3.4.** *Fix $m \in \mathbb{Z}$. Suppose $p > 0$ is very large relative to $m$. The map $\Phi$, and appropriate restrictions, fit into the following two diagrams, in which the unlabelled horizontal maps are the standard inclusions/projections and each vertical map is an isomorphism of chain complexes:*

$$\begin{tikzcd}
0 \arrow[r] & CFK^{\{i<0 \text{ or } j<0\}}(S^3,K,m) \arrow[r] \arrow[d, "\Phi"] & CFK^{\infty}(S^3,K,m) \arrow[r] \arrow[d, "\Phi"] & CFK^{\{i\geq 0, j\geq 0\}}(S^3,K,m) \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & CF^{-}(K_{-p},[m]) \arrow[r] & CF^{\infty}(K_{-p},[m]) \arrow[r] & CF^{+}(K_{-p},[m]) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & CFK^{\{\min(i,j)=0\}}(S^3,K,m) \arrow[r] \arrow[d, "\Phi"] & CFK^{\{i\geq 0, j\geq 0\}}(S^3,K,m) \arrow[r, "U"] \arrow[d, "\Phi"] & CFK^{\{i\geq 0, j\geq 0\}}(S^3,K,m) \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & \widehat{CF}(K_{-p},[m]) \arrow[r] & CF^{+}(K_{-p},[m]) \arrow[r, "U"] & CF^{+}(K_{-p},[m]) \arrow[r] & 0
\end{tikzcd}$$

*In the bottom diagram, the map $U : CFK^{\{i\geq 0, j\geq 0\}}(S^3,K,m) \to CFK^{\{i\geq 0, j\geq 0\}}(S^3,K,m)$ sends a generator $[\boldsymbol{x}, i, j]$ to $[\boldsymbol{x}, i-1, j-1]$.*

We have an analogous theorem for large positive surgeries. To state it, we first define a map $\Psi$ from $CF^{\infty}(K_p,[m])$ to $CFK^{\infty}(S^3,K,m)$. Changing notation from above, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})$ represent the surgery data $(K, \lambda_{-p})$. Now $m$ is a marked point in $\gamma_g$, and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a Heegaard triple for the cobordism $W$ from $K_{-p}$ to $S^3$. If $[\boldsymbol{x}, i]$ is a generator for $CF^{\infty}(K_p,[m])$, where $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, then

$$\Psi([\boldsymbol{x}, i]) = \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}, \psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y}) | \mu(\psi)=0, \frac{1}{2}\langle c_1(\underline{\mathfrak{s}}(\boldsymbol{y})), [\hat{F}]\rangle + n_w(\psi) - n_z(\psi) = m\}} \#\mathcal{M}(\psi) \cdot [\boldsymbol{y}, i - n_w(\psi), j - n_z(\psi)].$$

**Theorem 3.5.** *Fix $m \in \mathbb{Z}$. Suppose $p > 0$ is very large relative to $m$. The map $\Psi$, and appropriate restrictions, fit into the following two diagrams, in which the unlabelled horizontal maps are the standard inclusions/projections and each vertical map is an isomorphism of chain complexes:*

$$\begin{tikzcd}
0 \arrow[r] & CF^{-}(K_{p},[m]) \arrow[r] \arrow[d, "\Psi"] & CF^{\infty}(K_{p},[m]) \arrow[r] \arrow[d, "\Psi"] & CF^{+}(K_{p},[m]) \arrow[r] \arrow[d, "\Psi"] & 0 \\
0 \arrow[r] & CFK^{\{i<0, j<0\}}(S^3,K,m) \arrow[r] & CFK^{\infty}(S^3,K,m) \arrow[r] & CFK^{\{i\geq 0 \text{ or } j\geq 0\}}(S^3,K,m) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \widehat{CF}(K_{p},[m]) \arrow[r] \arrow[d, "\Psi"] & CF^{+}(K_{p},[m]) \arrow[r, "U"] \arrow[d, "\Psi"] & CF^{+}(K_{p},[m]) \arrow[r] \arrow[d, "\Psi"] & 0 \\
0 \arrow[r] & CFK^{\{\max(i,j)=0\}}(S^3,K,m) \arrow[r] & CFK^{\{i\geq 0 \text{ or } j\geq 0\}}(S^3,K,m) \arrow[r, "U"] & CFK^{\{i\geq 0 \text{ or } j\geq 0\}}(S^3,K,m) \arrow[r] & 0
\end{tikzcd}$$

Theorem 3.4 and Theorem 3.3 will be proved in Section 3.6; Theorem 3.5 is analogous.

As these theorems show, the question of how to compute $CFK^{\infty}(S^3,K)$ is crucial for computing the Heegaard Floer homology of large surgeries. In what follows, we outline an algorithm for performing the computation of $CFK^{\infty}(S^3,K)$. We then carry it out explicitly in a few concrete examples.

## 3.2 Generators of $CFK^{\infty}$

The first task will be to determine the generators of $CFK^{\infty}(S^3,K)$. Recall that an element $[\boldsymbol{x}, i, j]$, where $\boldsymbol{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$ and $i, j \in \mathbb{Z}$, represents a generator of $CFK^{\infty}(S^3,K)$ precisely when $\sigma([\boldsymbol{x}, i, j]) = \underline{\mathfrak{s}}(\boldsymbol{x}) + (i-j)PD[\mu] = \mathfrak{t}_0$. In particular, for elements with $i = 0$, the index $j$ is determined by $\boldsymbol{x}$, as a result of the equation $jPD[\mu] = \underline{\mathfrak{s}}(\boldsymbol{x}) - \mathfrak{t}_0$. Unfortunately, from this equation it is hard to see an explicit formula for $j$ given $\boldsymbol{x}$. Following [13], we will discuss an algorithm for computing $j$ which is based on the Fox calculus.

For $\boldsymbol{x} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$, let $j(\boldsymbol{x})$ be the unique integer such that $\sigma([\boldsymbol{x}, 0, j(\boldsymbol{x})]) = \mathfrak{t}_0$; in other words,

$$\underline{\mathfrak{s}}(\boldsymbol{x}) - j(\boldsymbol{x})PD[\mu] = \mathfrak{t}_0. \tag{2}$$

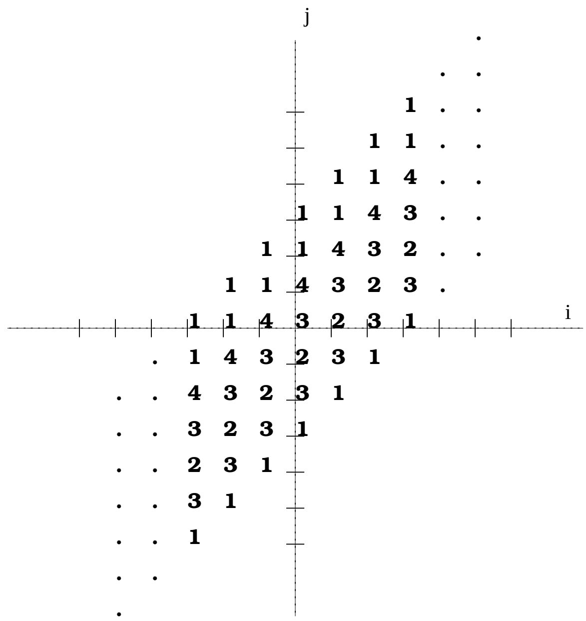


Figure 4: A grid representing the generators of $CFK^{\infty}(S^3, K)$, where $K$ is the $(3,4)$ torus knot.

We may list the generators of $CFK^{\{i=0\}}(S^3, K)$: they are triples of the form $[\boldsymbol{x}, 0, j(\boldsymbol{x})]$, where $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Applying the automorphism $U$, we obtain the following characterization of $CFK^{\infty}(S^3, K)$ as a group:

**Proposition 3.6.** *Let $K$ be a knot in $S^3$. Without regard to differentials (i.e. as an abelian group),*

$$CFK^{\infty}(S^3, K) = \bigoplus_{\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i \in \mathbb{Z}} \mathbb{Z} \cdot [\boldsymbol{x}, i, i + j(\boldsymbol{x})].$$

A good pictoral way to represent $CFK^{\infty}(S^3, K)$ is by drawing a grid. The horizontal axis is indexed by $i$, and the vertical axis is indexed by $j$. For each intersection point $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, mark the points on the line $\{(i, i + j(\boldsymbol{x})) | i \in \mathbb{Z}\}$. Then, at a point $(i, j)$ on the grid, place a number indicating how many times that point has been marked. Figure 4 shows this grid in the example of the $(3,4)$ torus knot.

We now turn to the issue of computing $j(\boldsymbol{x})$, given $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For this, we will use Fox calculus. We will describe an algorithm for attaching an integer $A(\boldsymbol{x})$ to each intersection point $\boldsymbol{x}$. The letter $A$ has been chosen since the Fox calculus techniques involved in computing $A(\boldsymbol{x})$ are identical to those used to compute the Alexander polynomial of $K$; we will call $A(\boldsymbol{x})$ the *Alexander grading* of $\boldsymbol{x}$. We will then show (Theorem 3.9) that, in fact, we have $A(\boldsymbol{x}) = j(\boldsymbol{x})$.

### 3.2.1 Fox calculus algorithm for computing Alexander gradings.

Choose a marked Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, m)$ for $K$ coming from a bridge presentation of $K$ as in Section 2.5.1. Pick arbitrary orientations of $\beta_1, \ldots, \beta_g$. Orient the $\alpha$ curves such that for each $\beta_i$, $1 \le i \le g-1$, the sum of the intersection numbers $\sum_j \beta_i \cdot \alpha_j$ is zero, and such that the short arc in $\alpha_i$ connects $w$ to $z$ (rather than $z$ to $w$). While this may not be possible in an arbitrary Heegaard diagram, it is clear from the form of the $\beta$ curves that it can be done when the diagram comes from a bridge presentation. Choose a "starting point" $Q_i$ on each $\beta_i$, away from intersections with the $\alpha$ curves. For a fixed $j$, the curve $\alpha_j$ intersects the $\beta$ curves in a number of points. Call these points $x_{j,1}$, $x_{j,2}$, $x_{j,3}$, $\ldots$. The second index here represents an arbitrary enumeration; it is not related to the index of any $\beta$ curve.

For each $\beta_i$, we will define a word $w_i$ in the formal letters $\alpha_j$ and their formal inverses. Begin at the starting point $Q_i$ and travel along $\beta_i$ in the direction of the chosen orientation. Upon reaching an intersection point of $\beta_i$ with a curve $\alpha_j$, add the letter $\alpha_j$ to the right side of the word if $\beta_i$ crosses $\alpha_j$

from right to left, and add $\alpha_j^{-1}$ if $\beta_i$ crosses from left to right. End the process after traversing $\beta_i$ once and arriving back at $Q_i$.

We now have $g$ words $\{w_1, \ldots, w_g\}$ in $g$ letters $\alpha_1, \ldots, \alpha_g$. Form the $g \times g$ matrix $M$ whose $(i,j)^{th}$ entry is the free derivative of the word $w_i$ with respect to the letter $\alpha_j$ ("free derivative" means the derivative in the sense of Fox calculus, which is computed slightly differently than a standard partial derivative). Now pick a new formal letter $t$ and set each $\alpha_j$ equal to $t$ (and each $\alpha_j^{-1}$ equal to $t^{-1}$). This procedure yields a $g \times g$ matrix $P$ whose entries are Laurent polynomials in $t$.

Consider the Laurent polynomial $P_{ij}$ corresponding to the free derivative of $w_i$ with respect to $\alpha_j$. Each term in $P_{ij}$ comes from an occurrence of $\alpha_j$ or $\alpha_j^{-1}$ in $w_i$, and each such occurence comes from a specific intersection point $x_{j,k}$ of $\alpha_j$ with $\beta_i$. Label the term with these two indices $j$ and $k$. For example, if $P_{ij} = 1 - t^{-2} + t^{-3}$, the terms might be labeled as $1_{4,2} - t^{-2}_{2,1} + t^{-3}_{1,4}$. It is clear that each pair of indices $(j,k)$ representing an intersection point occurs exactly once on a term in some $P_{ij}$.

Now take the determinant of $P$ using the permutation expansion. Make no cancellations, except that multiplication of any term by zero removes that term from the calculation. The result is a single Laurent polynomial in $t$; every term carries $g$ pairs of indices $(j_1, k_1), \ldots, (j_g, k_g)$. Since the indices $j_i$ must all be distinct, we have $\{j_1, \ldots, j_g\} = \{1, \ldots, g\}$. Thus, we may relabel $j_i$ as $i$. We will write the term carrying the indices $\{(1, k_1), \ldots, (g, k_g)\}$ as $t^d_{(1,k_1),\ldots,(g,k_g)}$.

A set of $g$ pairs of indices appearing on a term in $\det P$ comes from a set of $g$ intersection points $\{x_{1,k_1}, \ldots, x_{1,k_g}\}$. Furthermore, each pair of indices comes from a different row of $P$ and hence from a different $\alpha$ curve. Thus this set of intersection points corresponds to a point $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. In this way, each $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ appears exactly once in the expression for $\det P$.

**Definition 3.7.** For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, define $A'(\boldsymbol{x})$ to be the degree $d$ in $t$ of the corresponding term $t^d_{(1,k_1),\ldots,(g,k_g)}$ in $\det P$ described above.

The function $A'$ depends on various arbitrary choices such as the basepoints $Q_i$ in the $\beta$ curves. It turns out, though, that we can shift $A'$ by a constant independent of $\boldsymbol{x}$ to obtain a function $\boldsymbol{x} \mapsto A(\boldsymbol{x})$ which is independent of these choices. We first describe how to make this shift and then prove that $A(\boldsymbol{x}) = j(\boldsymbol{x})$.

To define the function $A$, note that if we allow ourselves to cancel terms in $\det P$, we get

$$\det P = \pm t^k \Delta(K), \tag{3}$$

for some $k \in \mathbb{Z}$, where $\Delta(K)$ is the symmetrized Alexander polynomial of $K$.

**Definition 3.8.** The Alexander grading $A(\boldsymbol{x})$ of $\boldsymbol{x}$ is $A'(\boldsymbol{x}) - k$, where $k$ is the unique integer satisfying $\det P = \pm t^k \Delta(K)$.

### 3.2.2 Proof that the Fox calculus algorithm works.

As promised, the Alexander grading $A(\boldsymbol{x})$ is equal to the index $j(\boldsymbol{x})$ defined earlier:

**Theorem 3.9.** *Let $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Then $A(\boldsymbol{x}) = j(\boldsymbol{x})$.*

*Proof.* Suppose $\boldsymbol{x} = \{x_{1,k_1}, \ldots, x_{g,k_g}\}$, and let $t^d_{(1,k_1),\ldots,(g,k_g)}$ be the corresponding term of $\det P$. We may write this term as $\Pi_{j=1}^g t^{d_j}_{g,d_g}$, where $t^{d_j}$ comes from taking the free derivative of some word $w_i$ as described above and setting every letter equal to $t$. Working out the mechanics of free derivatives, as well as the definition of the words $w_i$, we can calculate $d_j$ as follows. For each curve $\alpha_i$, we have chosen an orientation, so it makes sense to define a curve $\alpha_i'$ by pushing $\alpha_i$ slightly to the left with respect to its direction of travel. Then $d_j$ equals the number of intersections (with sign) of $\beta_j$ with the curves $\alpha_i'$ when traversing $\beta_j$ from the starting point $Q_j$ to the intersection point $x_{j,k_j}$. Hence $d = \sum_j d_j$ is the total number of intersections of $\beta$ curves with the $\alpha_i'$ curves, when the $\beta$ curves are traversed from the starting points $Q_j$ to the intersection points $x_{j,k_j}$.

Now suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are two points in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Let $t^d_{(1,k_1),\ldots,(g,k_g)}$ be the term in $\det P$ corresponding to $\boldsymbol{x}$, and let $t^{d'}_{(1,k_1'),\ldots,(g,k_g')}$ be the term corresponding to $\boldsymbol{y}$. The above computation tells us that $A(\boldsymbol{x}) - A(\boldsymbol{y}) = d - d'$ is the total number of intersections of $\beta$ curves with the $\alpha_i'$ curves, when $\beta_j$ is traversed from the point $x_{j,k_j'}$ to the point $x_{j,k_j}$.

Hence $A(\boldsymbol{x})-A(\boldsymbol{y})$ is a sum of intersection signs. We can break this sum up according to the $\alpha$ curves. For each $j$, let $B_j$ be the segment of $\beta_j$ connecting $x_{j,k'_j}$ to $x_{j,k_j}$. Then $A(\boldsymbol{x})-A(\boldsymbol{y})=\sum_i(\alpha'_i\cdot(\sum_j B_j))$, where $\cdot$ denotes the signed intersection number. Recall the curve $c_{\boldsymbol{x},\boldsymbol{y}}$ from Section 2.2.3. As long as $c_{\boldsymbol{x},\boldsymbol{y}}$ has no component along $\beta_g$, we have $\alpha'_i\cdot(\sum_j B_j)=\alpha'_i\cdot c_{\boldsymbol{x},\boldsymbol{y}}$, so

$$A(\boldsymbol{x})-A(\boldsymbol{y})=\sum_i \alpha'_i\cdot c_{\boldsymbol{x},\boldsymbol{y}}. \tag{4}$$

Note that $c_{\boldsymbol{x},\boldsymbol{y}}$ is only uniquely defined up to the addition of full $\alpha$ and $\beta$ curves. However, $\alpha'_i\cap\alpha_j=\emptyset$ for all $i,j$ and $(\sum_i\alpha'_i)\cdot\beta_j=(\sum_i\alpha_i)\cdot\beta_j=0$ for all $j\neq g$ by our orientation convention for the $\alpha$ curves. Hence, in equation 4, we can choose any representative for $c_{\boldsymbol{x},\boldsymbol{y}}$ in $\frac{H_1(\Sigma)}{\langle[\alpha_1],\ldots,[\alpha_g],[\beta_1],\ldots,[\beta_g]\rangle}$ so long as it has no component along $\beta_g$. In particular, we can take $c_{\boldsymbol{x},\boldsymbol{y}}=c'_{\boldsymbol{x},\boldsymbol{y}}$ minus any possible $\beta_g$ component, where $c'_{\boldsymbol{x},\boldsymbol{y}}=\partial D(\phi)$ for the unique $\phi\in\pi_2(\boldsymbol{x},\boldsymbol{y})$ with $n_w(\phi)=0$. We will denote this choice of $c_{\boldsymbol{x},\boldsymbol{y}}$ by $\delta$.

We have arrived at the equation $A(\boldsymbol{x})-A(\boldsymbol{y})=\sum_i(\alpha'_i\cdot\delta)$. Recall that the only $\alpha$ curve intersecting $\beta_g$ is $\alpha_g$. Hence, for $i\neq g$, we have $\alpha'_i\cdot\delta=\alpha'_i\cdot c'_{\boldsymbol{x},\boldsymbol{y}}$. But $c'_{\boldsymbol{x},\boldsymbol{y}}$ is zero in $H_1(\Sigma)$, so this intersection number vanishes. Thus most terms in our expression for $A(\boldsymbol{x})-A(\boldsymbol{y})$ cancel, and we are left with $A(\boldsymbol{x})-A(\boldsymbol{y})=\alpha'_g\cdot\delta$.

Now recall from Section 2.2.5 that we can compute the multiplicity of $\phi$ on a region by picking a point in that region and a path from $w$ to that point. In particular, to compute $n_z(\phi)$, we can pick a path $\gamma$ from $w$ to $z$. We choose $\gamma$ to be a segment of $\alpha'_g$ (or possibly a small isotopic translate), going the long way around and hence not intersecting $\beta_g$. Then $\alpha'_g\cdot\delta=\gamma\cdot\partial(D(\phi))$. From the computation in Section 2.2.5, a positive intersection of $\gamma$ with $\partial(D(\phi))$ with multiplicity $n$ adds $n$ to the coefficient in $D(\phi)$ of the next region, while a negative intersection with multiplicity $n$ subtracts $n$ from the coefficient. We start with coefficient 0 and end with $n_z(\phi)$, so $\gamma\cdot\partial(D(\phi))=n_z(\phi)$. We have now derived the equation $A(\boldsymbol{x})-A(\boldsymbol{y})=n_z(\phi)$.

Since $n_w(\phi)=0$, we may write this equation as $A(\boldsymbol{x})-A(\boldsymbol{y})=n_z(\phi)-n_w(\phi)$. Proposition 2.50 then becomes $\underline{\mathfrak{s}}(\boldsymbol{x})-\underline{\mathfrak{s}}(\boldsymbol{y})=(A(\boldsymbol{x})-A(\boldsymbol{y}))\cdot PD[\mu]$. But we also have $\underline{\mathfrak{s}}(\boldsymbol{x})-j(\boldsymbol{x})PD[\mu]=\mathfrak{t}_0$ and $\underline{\mathfrak{s}}(\boldsymbol{y})-j(\boldsymbol{y})PD[\mu]=\mathfrak{t}_0$. Therefore, $\underline{\mathfrak{s}}(\boldsymbol{x})-\underline{\mathfrak{s}}(\boldsymbol{y})=(j(\boldsymbol{x})-j(\boldsymbol{y}))PD[\mu]$. We may conclude that $A(\boldsymbol{x})-A(\boldsymbol{y})=j(\boldsymbol{x})-j(\boldsymbol{y})$. Rearranging terms, $A(\boldsymbol{x})-j(\boldsymbol{x})=A(\boldsymbol{y})-j(\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$.

Hence the mappings $\boldsymbol{x}\mapsto A(\boldsymbol{x})$ and $\boldsymbol{x}\mapsto j(\boldsymbol{x})$ only differ by a constant independent of $\boldsymbol{x}$. We are done if we can show this constant is zero. Indeed, another way of saying that $A$ and $j$ differ by a constant is to say that for some $l\in\mathbb{Z}$,

$$\det P=\pm t^l\sum_{\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}}(-1)^{\operatorname{sgn}\boldsymbol{x}}t^{j(\boldsymbol{x})}.$$

Here, $\operatorname{sgn}\boldsymbol{x}$ denotes the $\mathbb{Z}/2$-grading of $\boldsymbol{x}$, which is well-defined since we chose orientations on the $\alpha$ and $\beta$ curves. The above equation holds because $\operatorname{sgn}\boldsymbol{x}$ agrees with the sign on the term corresponding to $\boldsymbol{x}$ in $\det P$. Note that we may write the right-hand side of this equation as $\pm t^l\sum_{d\in\mathbb{Z}}t^d\chi(\widehat{CFK}(S^3,K,d))$.

Now compare this formula with Equation 3, which says $\det P=\pm t^k\Delta(K)$ for some $k$. We are done if we can show $k=l$, since $A$ was defined by shifting degrees in $\det P$ by $k$, while $l$ was defined as the shift in degree from $\det P$ to the function $\boldsymbol{x}\mapsto j(\boldsymbol{x})$. Putting our formulas together, we have

$$\Delta(K)=\pm t^{l-k}\sum_{d\in\mathbb{Z}}t^d\chi(\widehat{CFK}(S^3,K,d)),$$

and we want $l-k=0$.

Since the highest and lowest degrees of $\Delta(K)$ are symmetric about zero, it will suffice to show the same is true for the right-hand side of the above equation. But $\chi(\widehat{CFK}(S^3,K,d))=\chi(\widehat{HFK}(S^3,K,d))$, and by Proposition 2.57, we have the conjugation symmetry $\widehat{HFK}(S^3,K,d)=\widehat{HFK}(S^3,K,-d)$. Thus, the highest and lowest degrees of the polynomial $\sum_{d\in\mathbb{Z}}t^d\chi(\widehat{CFK}(S^3,K,d))$ are symmetric about zero. Hence $l-k=0$, completing the proof.

□

*Remark* 3.10. Note that since $\beta_g$ intersects $\alpha_g$ once and intersects no other $\alpha$ curves, the bottom row of the matrix $P$ will always look like $(0,\ldots,0,\pm1)$. This fact has a few consequences. First of all, $\det P$ is equal to the determinant of the submatrix $(P_{ij})_{i,j\leq g-1}$. Second of all, points $\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$, which *a*

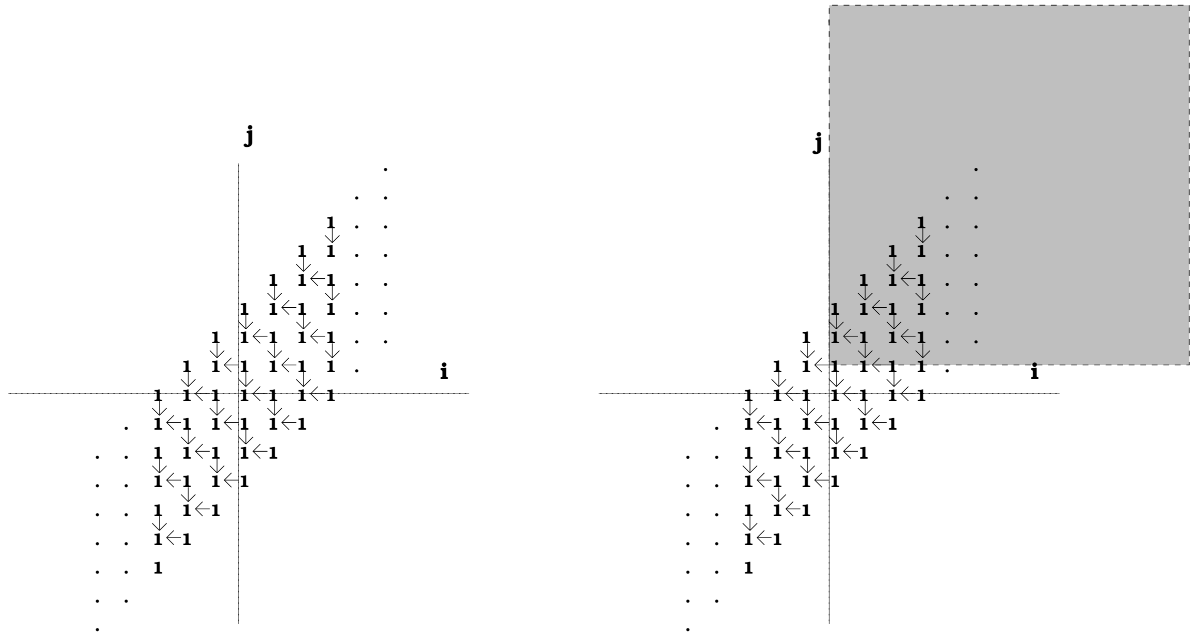

Figure 5: A grid representing $CFK^\infty(S^3, K)$, where $K$ is the $(2,7)$ torus knot. Since no differentials preserve both $i$ and $j$ in this example, the grid also represents $CFK^\infty_s(S^3, K)$. The right-hand side represents $CFK^{\{i\geq 0, j\geq 1\}}(S^3, K)$.

*priori* are described by $g$ pairs of indices $\{(1,k_1),\ldots,(g,k_g)\}$, may actually be described by the $g-1$ pairs of indices $\{(1,k-1),\ldots,(g-1,k_{g-1})$. We will reflect this fact in our notation. Hence, we will often take $\boldsymbol{x}$ to be synonymous with $\{x_{1,k_1},\ldots,x_{g-1,k_{g-1}}\}$, and a term of $\det P$ will be labelled like $t^d_{(1,k_1),\ldots,(g-1,k_{g-1})}$.

Furthermore, when $g=2$, this convention means that a point $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ may be described by a single pair of indices: $\boldsymbol{x} = \{x_{1,k_1}\}$. In this case, we will drop the 1 from the notation, and simply write $\boldsymbol{x} = x_k$. A term of $\det P$ will be labelled like $t^d_k$ for some $k$. This change will simplify the notation for the first two examples we consider, since both have $g=2$. The final example, though, will have $g=3$, and so this further simplification will not apply.

## 3.3 Differentials in $CFK^\infty$

### 3.3.1 Symmetries and reduction to $CFK^{\{i=0\}}(S^3, K)$.

The other half of the picture consists of finding the differentials in $CFK^\infty(S^3, K)$. We will indicate these differentials with arrows in the grid diagram. An arrow will stand for a single disk $\phi$ with $\mu(\phi) = 1$ and $\#\overline{\mathcal{M}}(\phi) = \pm 1$. In our examples, these are the only differentials. Figure 5 illustrates the complete grid, with differentials, in the case of the $(2,7)$ torus knot. It also shows the portion of the grid which computes $HF^+(K_{-p}, [-1])$ for large $p$, by Theorem 3.3.

Since $CFK^\infty(S^3, K)$ has many generators, computing the differentials may seem a daunting task. The $i \leftrightarrow j$ symmetry, though, will simplify things considerably. This symmetry does not hold on the level of $CFK^\infty(S^3, K)$, as Figure 4 demonstrates. However, we have two filtrations on $CFK^\infty(S^3, K)$. Define $CFK^\infty_s(S^3, K)$ to be the complex obtained from $CFK^\infty(S^3, K)$ by cancelling all differentials which preserve both the $i$-filtration and the $j$-filtration.

**Proposition 3.11.** *The complex $CFK^\infty_s(S^3, K)$ is symmetric under the interchange of $i$ and $j$.*

As is typical with filtered complexes, cancelling the differentials of $CFK^\infty(S^3, K)$ step-by-step yields the homology of the complex. Thus, reducing to $CFK^\infty_s(S^3, K)$ first and then taking homology produces $HFK^\infty(S^3, K)$. More to the point, note that to compute $H_*(CFK^{\{i\geq 0, j\geq -m\}}(S^3, K))$, we can cancel the filtration-preserving differentials at the very beginning, before getting rid of $CFK^{\{i<0 \text{ or } j<-m\}}(S^3, K)$. In other words, $H_*(CFK^{\{i\geq 0, j\geq -m\}}(S^3, K)) = H_*(CFK_s^{\{i\geq 0, j\geq -m\}}(S^3, K))$.

We have reduced our work to computing the complex $CFK_s^\infty(S^3, K)$. The automorphism $U$ will simplify the situation even further. If we can compute $CFK_s^{\{i=0\}}(S^3, K)$, then by applying powers of $U$ we can reconstruct all differentials in $CFK_s^\infty(S^3, K)$ which preserve $i$. Proposition 3.11 then gives us the differentials which preserve $j$. But in our examples, the homological grading will ensure that no differentials can decrease both $i$ and $j$. Hence we will have rebuilt all of $CFK_s^\infty(S^3, K)$.

In summary, we will analyze our examples by first computing the complex $CFK^{\{i=0\}}(S^3, K)$. We will then cancel the differentials which preserve $j$, producing $CFK_s^{\{i=0\}}(S^3, K)$. Applying $U$ and using Proposition 3.11 as just described, we will obtain $CFK_s^\infty(S^3, K)$. Finally, taking homology of the appropriate portion of this complex gives $HF^+$ of large surgeries on $K$.

### 3.3.2 Maslov indices.

Generators of $CFK^{\{i=0\}}(S^3, K)$ are points $[\boldsymbol{x}, 0, j]$ where $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $j = A(\boldsymbol{x})$. Since $j$ is determined entirely by $\boldsymbol{x}$, we will often just refer to the generators of the complex as points $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$.

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are generators of $CFK^{\{i=0\}}(S^3, K)$, then finding the $\boldsymbol{y}$-component of $\partial \boldsymbol{x}$ amounts to finding a domain $D$ in $\Sigma$ representing an element $\phi$ of $\pi_2(\boldsymbol{x}, \boldsymbol{y})$, computing $\mu(\phi)$, and (when this index is 1 and $n_w(D)$ is zero) counting points in the moduli space $\overline{\mathcal{M}}(\phi)$. We will see that, given $D$, there is an easy method of computing the Maslov index of the associated disk $\phi$. When $\mu(\phi) = 1$, it still may be very difficult to count the points in $\overline{\mathcal{M}}(\phi)$. Luckily, though, for certain types of domains, there are theorems saying that $\#(\overline{\mathcal{M}}(\phi))$ is always $\pm 1$, and in our examples we will rarely need to consider types other than these.

First, we will deal with the Maslov index. Let $D = \sum n_i \sigma_i$ be a domain in $\Sigma$ representing a homotopy class $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Recall from Section 2.2.5 that the regions $\sigma_i$ are the components of $\Sigma \setminus \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$. We will compute the Maslov index of $\phi$ in terms of the Euler measure of $D$ and the multiplicities of $D$ at $\boldsymbol{x}$ and $\boldsymbol{y}$, which we will now define:

**Definition 3.12.** For one of the regions $\sigma_i$ defined above, the Euler measure of $\sigma_i$ is defined to be the Euler characteristic of $\sigma_i$ minus $\frac{1}{4}$ times the number $k$ of corners of $\sigma_i$; concisely,

$$e(\sigma_i) := \chi(\sigma_i) - \frac{1}{4}k.$$

For a general domain $D = \sum n_i \sigma_i$, we define $e(D) := \sum n_i e(\sigma_i)$.

For the multiplicities of $D$ at $\boldsymbol{x}$ and $\boldsymbol{y}$, suppose $\boldsymbol{x} = \{x_1, \ldots, x_g\}$ and $\boldsymbol{y} = \{y_1, \ldots, y_g\}$. Since the points $x_i$ and $y_i$ are on the $\alpha$ and $\beta$ curves which form the boundary of $D$, we need a convention for the multiplicity of $D$ at these points. We do it by averaging, as follows:

**Definition 3.13.** Suppose $x$ is in the intersection of the $\alpha$ and $\beta$ curves, and let $D$ be a domain in $\Sigma$ as above. Let $\sigma_{i_1}, \ldots, \sigma_{i_4}$ be the four regions with $x$ as a corner. Define $n_x(D) = \frac{1}{4}\sum_{k=1}^4 n_{i_k}$. For a point $\boldsymbol{x} = \{x_1, \ldots, x_g\} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, define $n_{\boldsymbol{x}}(D) = \sum_{i=1}^g n_{x_i}(D)$.

With these definitions in place, we have the following characterization of the Maslov index:

**Proposition 3.14.** *If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, and $D = D(\phi)$ is the domain of $\phi$, then*

$$\mu(\phi) = e(D) + n_{\boldsymbol{x}}(D) + n_{\boldsymbol{y}}(D).$$

*Proof.* See [3]. □

### 3.3.3 Some domains with with $\#\overline{\mathcal{M}}(\phi) = \pm 1$.

Now we will list a few types of domains with Maslov index 1 with the property that for the corresponding homotopy classes $\phi$, we always have $\#\overline{\mathcal{M}}(\phi) = \pm 1$. A more thorough discussion of these facts can be found in the Appendix to [13].

**Proposition 3.15.** *Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and suppose $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ is a disk whose domain $D$ is a $2g$-gon, i.e. a topologically trivial embedded disk whose boundary consists of $2g$ arcs along the $\alpha$ and $\beta$ curves connecting each $x_i$ to a $y_j$. Then $\mu(\phi) = 1$ and $\#\overline{\mathcal{M}}(\phi) = \pm 1$.*

**Proposition 3.16.** *Let $\boldsymbol{x}^+ = \{x_1, \dots, x_{g-1}, x^+\}$ and $\boldsymbol{x}^- = \{x_1, \dots, x_{g-1}, x^-\}$ be two elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ differing in only one point. Suppose $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ is a disk whose domain $D$ satisfies:*

*(a) $\partial(D)$ is supported entirely on $\alpha_g$ and $\beta_g$.*

*(b) Topologically, $D$ is a collection of disks joined by at most $g-2$ handles.*

*Then $\mu(\phi) = 1$ and $\#\overline{\mathcal{M}}(\phi) = \pm 1$.*

## 3.4 Absolute grading of $CFK^{\infty}$

With these tools in hand, we can begin looking for differentials in $CFK^{\{i=0\}}(S^3, K)$. Once we have found them all, the final required piece is to identify the homological grading of each generator. To see the relative gradings, it will usually suffice to note that any differential, in any of the complexes under consideration, decreases homological degree by 1, and $U$ decreases homological degree by 2. Proposition 2.57 also tells us how the grading of generators in one Spin$^c$ structure relates to the grading in the conjugate Spin$^c$ structure. However, we need something to fix the absolute gradings. The solution is to view $CFK^{\{i=0\}}(S^3, K)$ as the $E^0$ term of a spectral sequence computing $\widehat{HF}(S^3) \simeq \mathbb{Z}$. The requirement that this $\mathbb{Z}$ have homological grading 0 fixes the homological gradings of $CFK^{\{i=0\}}(S^3, K)$ once the relative gradings are known. The following proposition summarizes the situation.

**Proposition 3.17.** *There exists a spectral sequence converging to $\widehat{HF}(S^3) \simeq \mathbb{Z}$ whose $E^1$ term is equal to $\widehat{HFK}(S^3, K)$.*

*Proof.* Recall from Section 2.5.2 that the index $j$ gives $\widehat{CF}(S^3) \simeq CFK^{\{i=0\}}(S^3, K)$ the structure of a filtered complex. We may thus view the complex as the $E^0$-term of a spectral sequence converging to $\widehat{HF}(S^3) \simeq \mathbb{Z}$, as shown below:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & CFK_d^{\{i=0,j=1\}}(S^3,K) \arrow[r] \arrow[dr] \arrow[ddr] & CFK_{d-1}^{\{i=0,j=1\}}(S^3,K) \arrow[r] \arrow[dr] \arrow[ddr] & CFK_{d-2}^{\{i=0,j=1\}}(S^3,K) & \cdots \\
\cdots & CFK_d^{\{i=0,j=0\}}(S^3,K) \arrow[r] \arrow[dr] & CFK_{d-1}^{\{i=0,j=0\}}(S^3,K) \arrow[r] \arrow[dr] & CFK_{d-2}^{\{i=0,j=0\}}(S^3,K) & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The subscripts denote absolute homological degrees. Since $\widehat{CFK}(S^3, K)$ is the graded complex associated to the $j$-filtration on $CFK^{\{i=0\}}(S^3, K)$, we may write the $E^1$ term as

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots & \widehat{HFK}_d(S^3,K,1) \arrow[dr] \arrow[ddr] & \widehat{HFK}_{d-1}(S^3,K,1) \arrow[dr] \arrow[ddr] & \widehat{HFK}_{d-2}(S^3,K,1) & \cdots \\
\cdots & \widehat{HFK}_d(S^3,K,0) \arrow[dr] & \widehat{HFK}_{d-1}(S^3,K,0) \arrow[dr] & \widehat{HFK}_{d-2}(S^3,K,0) & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Suppressing mention of the differentials and absolute gradings, we may write this $E^1$ term as

$$\bigoplus_{j\in\mathbb{Z}} \widehat{HFK}(S^3,K,j) = \widehat{HFK}(S^3,K),$$

and hence we have proved Proposition 3.17. $\square$

## 3.5 Examples

### 3.5.1 Destabilization of doubly-pointed Heegaard diagrams

Fix a marked Heegaard diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},m)$ for $K$ and let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$ be the corresponding two-pointed Heegaard diagram. For computations, the following fact will be helpful.

*Remark* 3.18. Suppose $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$ is a two-pointed Heegaard diagram such that one of the curves in $\boldsymbol{\beta}$, say $\beta_g$, intersects only one $\boldsymbol{\alpha}$ curve, say $\alpha_g$, in one point. Let $\Sigma'$ be the surface obtained from $\Sigma$ by a surgery which removes $\beta_g$. Let $\boldsymbol{\alpha}_0=\boldsymbol{\alpha}\setminus\{\alpha_g\}$ and let $\boldsymbol{\beta}_0=\boldsymbol{\beta}\setminus\{\beta_g\}$, viewed as curves in $\Sigma'$. There is an obvious bijection between $\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$ and $\mathbb{T}_{\boldsymbol{\alpha}_0}\cap\mathbb{T}_{\boldsymbol{\beta}_0}$.

**Lemma 3.19.** *For a suitable choice of complex structure on $\Sigma'$, the obvious bijection between $\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$ and $\mathbb{T}_{\boldsymbol{\alpha}_0}\cap\mathbb{T}_{\boldsymbol{\beta}_0}$ gives an isomorphism of chain complexes $CF^\infty(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)\simeq CF^\infty(\Sigma',\boldsymbol{\alpha}_0,\boldsymbol{\beta}_0,w',z')$, where $w'$ and $z'$ are points in $\Sigma'$ corresponding to $w$ and $z$ in $\Sigma$.*

For the idea of the proof of Lemma 3.19, see the proof of Theorem 1.1 in Section 10 of [11].

In the examples we will consider, the conditions of the lemma will always hold, with $\beta_g$ equal to the meridian of the knot. Thus, to compute differentials in $CFK^\infty(S^3,K)$, it will suffice to count holomorphic disks in the destabilized diagram, an easier task because the genus has been reduced by one.

### 3.5.2 The left-handed trefoil.

Now we will carry out the above procedure in a few examples. The first is the left-handed trefoil. We use the Heegaard diagram derived in Section 2.5.1; see Figure 6. The generators of $CFK^\infty(S^3,K)$ correspond to the points $x_1$, $x_2$, and $x_3$ in the diagram. Our first task is to compute their Alexander gradings. To obtain the matrix $P$, we first compute the words $w_i$ and their free derivatives, for $1\le i\le g-1$. Here, $g=2$, so we need only consider $w_1$.

| Word | Expression |
|---|---|
| $w_1$ | $\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}$ |
| $\partial_1 w_1$ | $\alpha_2-\alpha_2\alpha_1\alpha_2\alpha_1^{-1}-\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}$ |

Replacing all the letters $\alpha_i$ by $t$, we get the following matrix, where $*$ denotes an entry irrelevant for the determinant:

$$P=\begin{pmatrix} t-t^2-1 & * \\ 0 & 1\end{pmatrix}$$

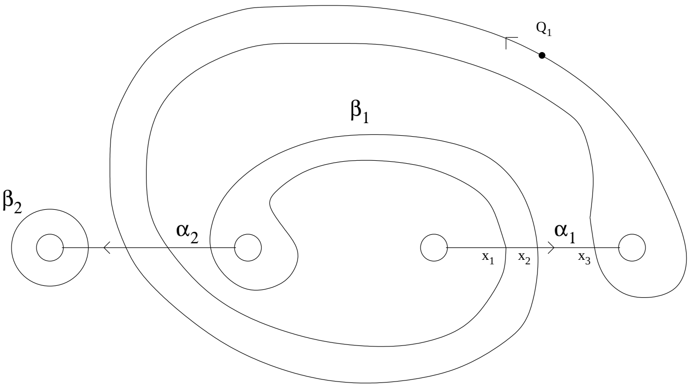


Figure 6: This is the Heegaard diagram for the trefoil before it has been destabilized.

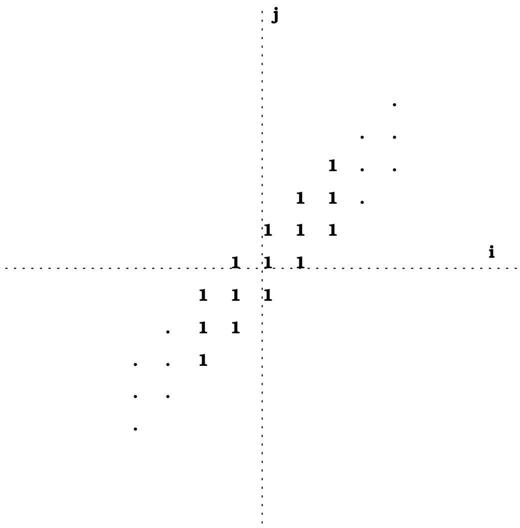


Figure 7: $CFK^\infty(S^3, K)$. A 1 indicates a copy of $\mathbb{Z}$.

Thus, $\det P = t - t^2 - 1$. At this point, we examine the terms to see how they should be labelled; we get $\det P = t_2 - t_1^2 - 1_3$. Thus, before symmetrizing, we would have $A(x_1) = 2$, $A(x_2) = 1$, and $A(x_3) = 0$. After symmetrizing, we get the true Alexander gradings, which are

| Generator $\boldsymbol{x}$ | Alexander grading $A(\boldsymbol{x})$ |
|---|---|
| $x_1$ | 1 |
| $x_2$ | 0 |
| $x_3$ | $-1$ |

At this point we have identified the generators of $CFK^\infty(S^3, K)$; Figure 7 depicts them in grid form.

Now we want to identify the differentials in $CFK^\infty(S^3, K)$, or equivalently just the differentials in $CFK^{\{i=0\}}(S^3, K)$. For this purpose, it will be easier to work with the destabilized diagram shown in Figure 8.

There is one obvious differential from $x_1$ to $x_2$, with coefficient $\pm 1$. In fact, this is the only differential in $CFK^\infty(S^3, K)$. The situation is summarized in Figure 9. From this, we can first conclude that $\widehat{HFK}(S^3, K, d) = CFK^{\{i=0, j=d\}}(S^3, K)$, so we have

$$\widehat{HFK}(S^3, K, d) = \begin{cases} \mathbb{Z} & \text{if } -1 \leq d \leq 1; \\ 0 & \text{otherwise.} \end{cases}$$

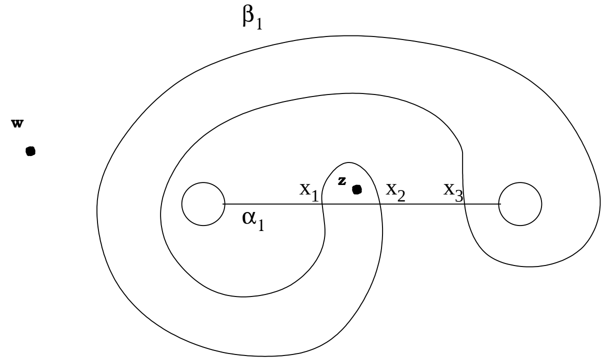


Figure 8: This diagram is the result of destabilization.

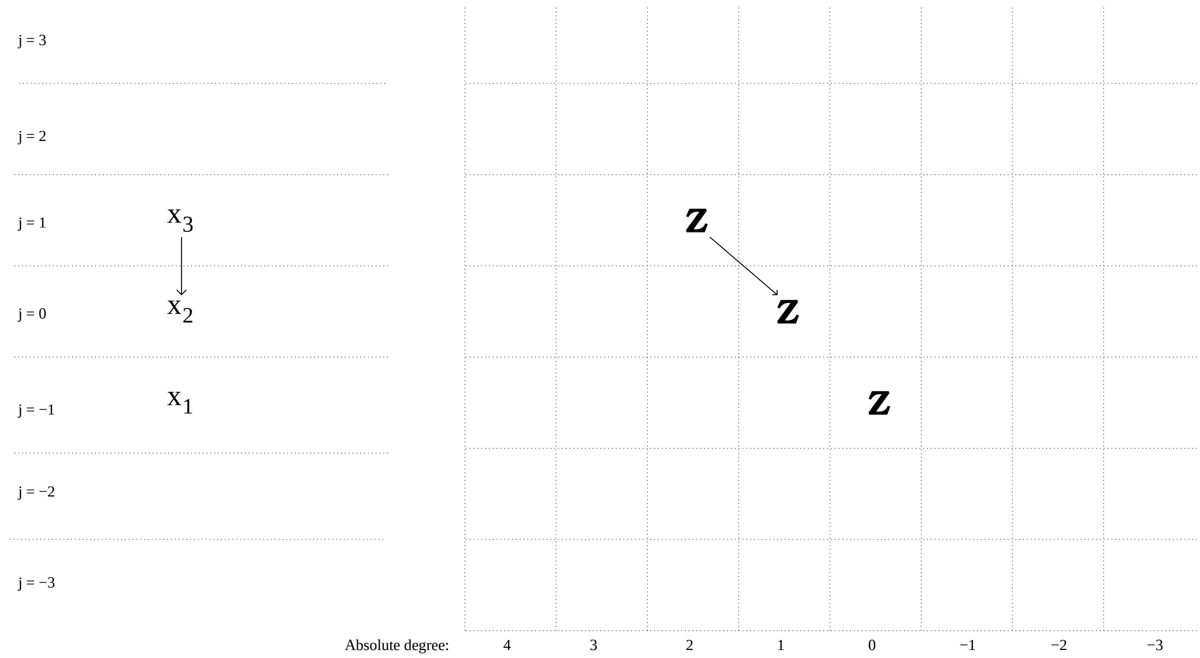


Figure 9: The $E^0$ term of the spectral sequence for the left-handed trefoil. The symbol $x_i$ stands for the group $\mathbb{Z}\cdot x_i$, and arrows indicate a differential with coefficient $\pm 1$. Since there are no filtration-preserving differentials, this figure also represents the $E^1$ term.

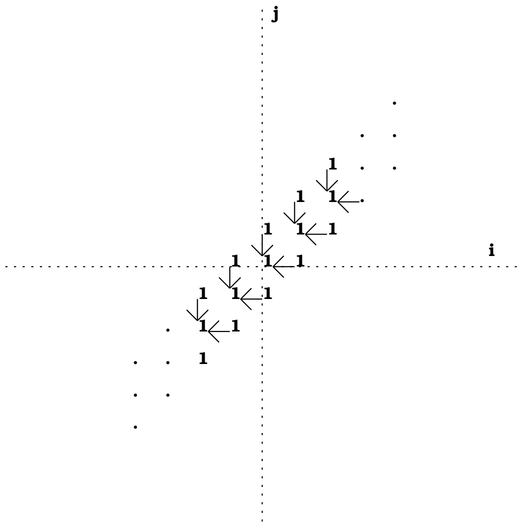


Figure 10: $CFK^\infty(S^3, K)$, showing both generators and differentials.

Furthermore, since the homology of this complex computes $\widehat{HF}(S^3) = \mathbb{Z}$, which is localized in absolute degree 0, we see that the absolute degree of the generator $x_3$ is 0.

Using the symmetries of $CFK^\infty(S^3, K)$, we can construct a grid picture of the whole complex; see Figure 10. From this grid, and the fact that the automorphism $U$ of $CFK^\infty(S^3, K)$ decreases absolute degree by 2 while any differential decreases it by 1, we see that $\deg x_1 = 2$ and $\deg x_2 = 1$.

The complex depicted in Figure 10 tells us how to compute $HF^+(K_{-p}, [m])$, with absolute degrees, for any $m$. We just take the homology of the appropriate portion of the complex and apply Theorem 3.3. The absolute gradings also come from Theorem 3.3.

#### 3.5.3 The $(2, 7)$ torus knot.

Our next example is the $(2, 7)$ torus knot; $K$ will now denote this knot. Figure 11 shows a Heegaard diagram for $K$ derived from its Schubert normal form. The generators correspond to $x_1, \ldots, x_7$. The table below shows $w_1$ and its free derivative:

| Word | Expression |
|---|---|
| $w_1$ | $\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}$ |
| $\partial_1 w_1$ | $\alpha_2 + \alpha_2\alpha_1\alpha_2 + \alpha_2\alpha_1\alpha_2\alpha_1\alpha_2 - \alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1^{-1}$ |
| | $-\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1} - \alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}$ |
| | $-\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_2^{-1}\alpha_1^{-1}$ |

Replacing the letters $\alpha_i$ by $t$, we get

$$P = \begin{pmatrix} t + t^3 + t^5 - t^6 - t^4 - t^2 - 1 & * \\ 0 & 1 \end{pmatrix}$$

and hence $\det P = t + t^3 + t^5 - t^6 - t^4 - t^2 - 1$. Labelling the terms with their corresponding intersection points, this is $t_6 + t_4^3 + t_2^5 - t_1^6 - t_3^4 - t_5^2 - 1_7$. Thus, after symmetrizing, we have the following table:

| Generator $\boldsymbol{x}$ | Alexander grading $A(\boldsymbol{x})$ |
|---|---|
| $x_1$ | 3 |
| $x_2$ | 2 |
| $x_3$ | 1 |
| $x_4$ | 0 |
| $x_5$ | $-1$ |
| $x_6$ | $-2$ |
| $x_7$ | $-3$ |

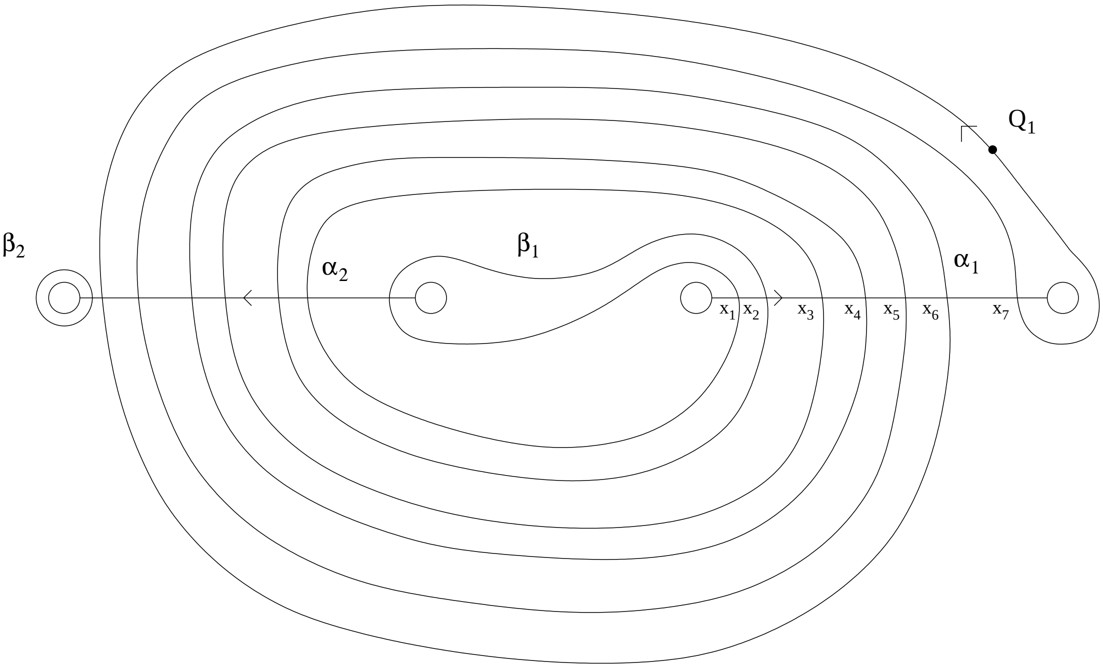


Figure 11: A Heegaard diagram for the $(2,7)$ torus knot.

Now we can write down the generators of $CFK^{\infty}(S^3,K)$ in grid form; they appear in Figure 12.

Our next task is to find the differentials in $CFK^{\{i=0\}}(S^3,K)$. We will destabilize the diagram, producing Figure 13. This time, there are three differentials in $CFK^{\{i=0\}}(S^3,K)$. They go from $x_1$ to $x_2$, from $x_3$ to $x_4$, and from $x_5$ to $x_6$, each with coefficient $\pm 1$. Figure 14 illustrates the complex $CFK^{\{i=0\}}(S^3,K)$ with differentials added.

Again, there are no filtration-preserving differentials, so

$$\widehat{HFK}(S^3,K,d) = CFK^{\{i=0,j=d\}}(S^3,K) = \begin{cases} \mathbb{Z} & \text{if } -3 \leq d \leq 3; \\ 0 & \text{otherwise.} \end{cases}$$

The homology of $CFK^{\{i=0\}}(S^3,K)$ is $\mathbb{Z}$, generated by $x_7$. Hence $x_7$ has absolute degree 0. Furthermore, we can use the symmetries of $CFK^{\infty}(S^3,K)$ to identify the differentials in this complex; they are shown in the grid in Figure 15, which reproduces Figure 5 from earlier. As with the trefoil, this grid, plus the fact that $U$ decreases absolute degree by 2 while any differential decreases it by 1, pins down the absolute degree of each $x_i$. We have $\deg x_1 = 7$, $\deg x_2 = 6$, $\deg x_3 = 5$, and so on; in general, $\deg x_i = 8 - i$. Cutting out the appropriate part of this diagram and taking homology produces $HF^+(K_{-p},[m])$ by Theorem 3.3.

#### 3.5.4 The $(3,4)$ torus knot.

Our final example is the $(3,4)$ torus knot; $K$ will now denote this knot. A Heegaard diagram for $K$ is shown in Figure 16. The diagram arises from a bridge presentation for $K$, although to save space we will not show the details. In this example, the genus of the diagram is 3 rather than 2, so we might expect additional complications to arise. By our previous convention, we should label the intersection points as $x_{1,1},\dots,x_{1,6}$ and $x_{2,1},\dots,x_{2,5}$. However, to get rid of an index, we will define $x_k := x_{1,k}$ and $y_k := x_{2,k}$. Then an intersection point $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ will correspond to a set of two points $(x_k, y_l)$ where $1 \leq k \leq 6$ and $1 \leq l \leq 5$.

The following table lists the words $w_i$ and their formal derivatives:

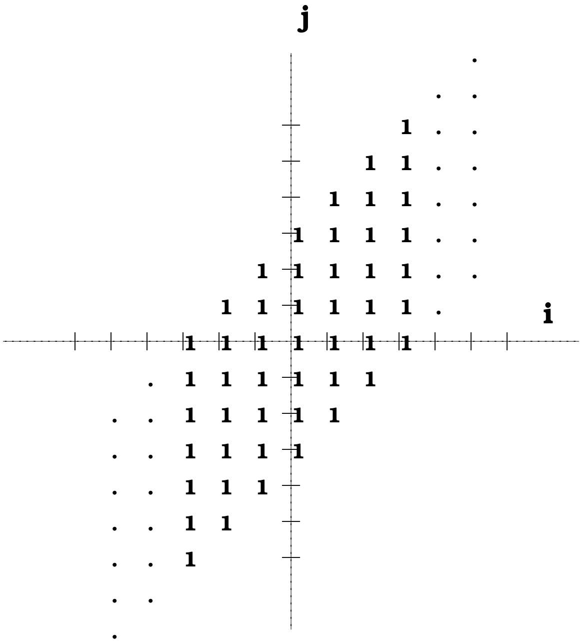


Figure 12: The generators of $CFK^{\infty}(S^3, K)$.

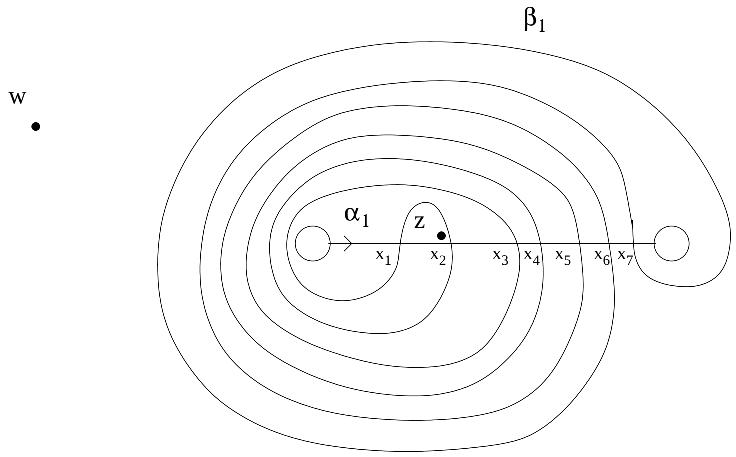


Figure 13: The destabilized Heegaard diagram for the $(2, 7)$ torus knot.

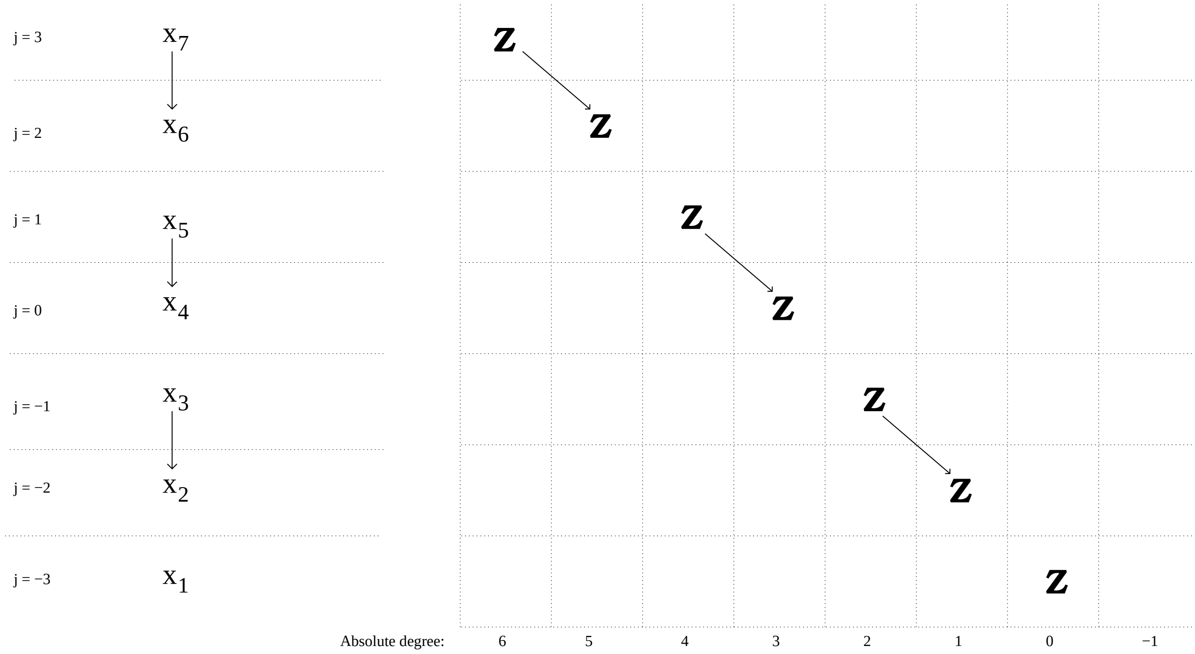


Figure 14: The $E^0$ term of the spectral sequence in the case of the $(2,7)$ torus knot. As before, the symbol $x_i$ stands for the group $\mathbb{Z} \cdot x_i$, and arrows indicate a differential with coefficient $\pm 1$. The figure also represents the $E^1$ term.

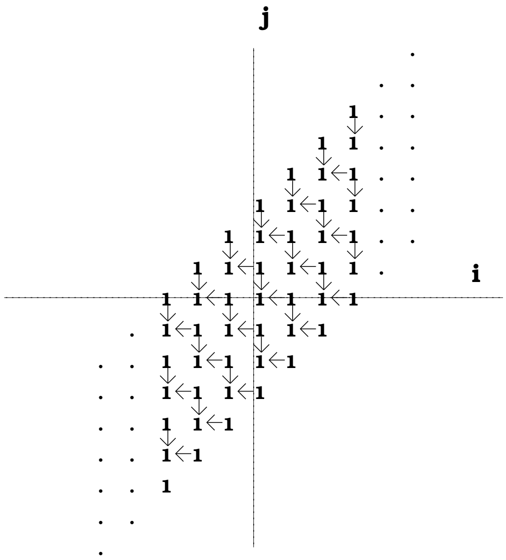


Figure 15: Differentials in $CFK^{\infty}(S^3, K)$.

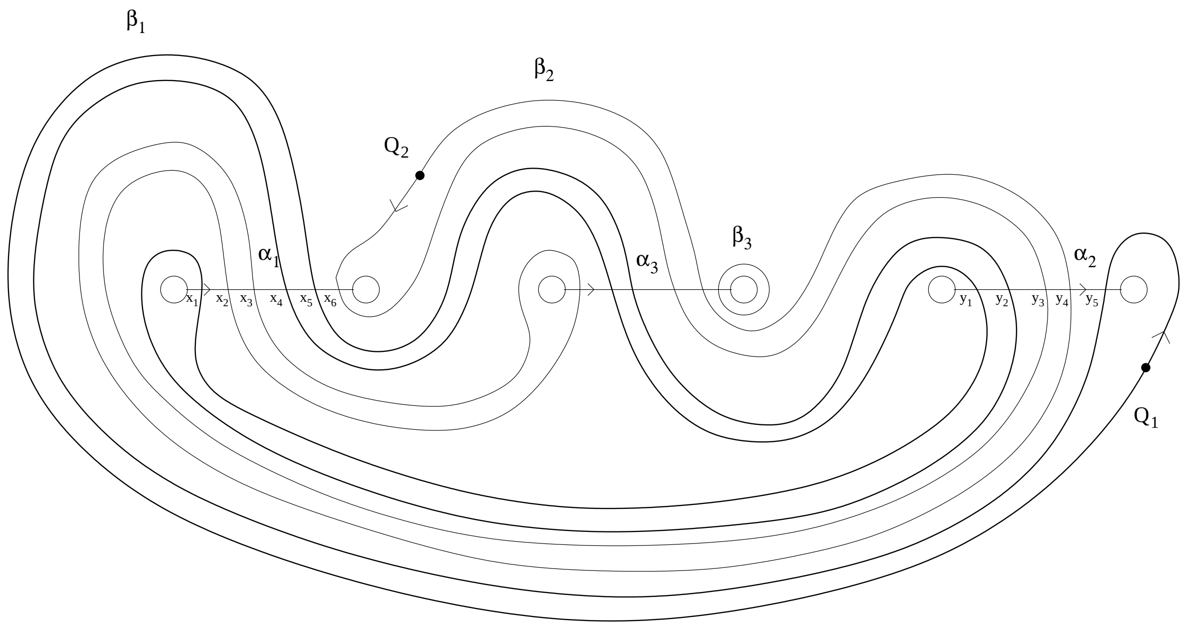


Figure 16: A Heegaard diagram for the $(3, 4)$ torus knot.

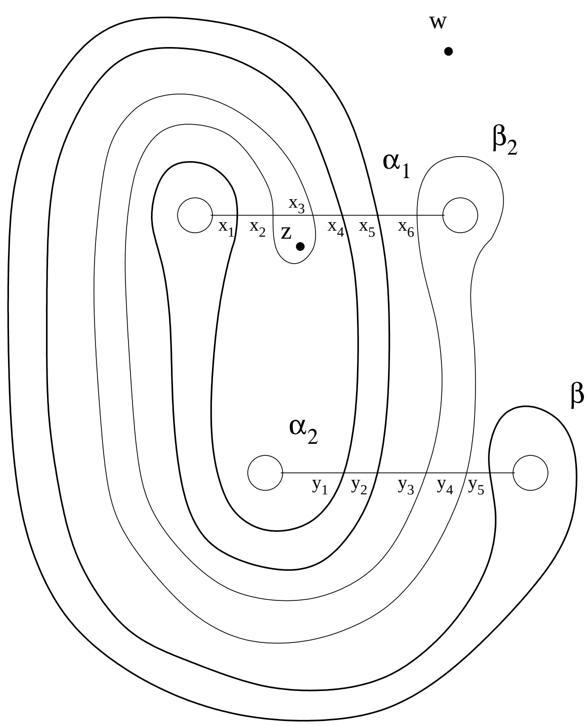


Figure 17: The destabilized Heegaard diagram for the $(3, 4)$ torus knot.

| Word | Expression |
|---|---|
| $w_1$ | $\alpha_2^{-1}\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1\alpha_2\alpha_3\alpha_1$ |
| $\partial_1 w_1$ | $-\alpha_2^{-1}\alpha_1^{-1} + \alpha_2^{-1}\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1} + \alpha_2^{-1}\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1\alpha_2\alpha_3$ |
| $\partial_2 w_1$ | $-\alpha_2^{-1} - \alpha_2^{-1}\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1} + \alpha_2^{-1}\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1$ |
| $w_2$ | $\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_3\alpha_1\alpha_2\alpha_3$ |
| $\partial_1 w_2$ | $-\alpha_1^{-1} - \alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1^{-1} + \alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_3$ |
| $\partial_2 w_2$ | $-\alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1} + \alpha_1^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_1^{-1}\alpha_3\alpha_1$ |

Thus,

$$P = \begin{pmatrix} -t^{-2}+t^{-4}+t^{-1} & -t^{-1}-t^{-4}+t^{-3} & * \\ -t^{-1}-t^{-4}+t^{-3} & -t^{-3}+t^{-2} & * \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\det P = (-t_{x_4}^{-2} + t_{x_1}^{-4} + t_{x_5}^{-1})(-t_{y_3}^{-3} + t_{y_4}^{-2}) - (-t_{x_6}^{-1} - t_{x_2}^{-4} + t_{x_3}^{-3})(-t_{y_5}^{-1} - t_{y_1}^{-4} + t_{y_2}^{-3}).$$

Because we modified the index convention, we have taken the liberty of labelling terms with the actual intersection points rather than with the corresponding indices. Also, to avoid a large mess, we have not expanded $\det P$. It should be clear how to extract the Alexander gradings for the various elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ from this formula; the table below lists them.

| Generator $\boldsymbol{x}$ | Alexander grading $A(\boldsymbol{x})$ |
|---|---|
| $(x_1, y_3)$ | $-2$ |
| $(x_1, y_4)$ | $-1$ |
| $(x_2, y_1)$ | $-3$ |
| $(x_2, y_2)$ | $-2$ |
| $(x_2, y_5)$ | $0$ |
| $(x_3, y_1)$ | $-2$ |
| $(x_3, y_2)$ | $-1$ |
| $(x_3, y_5)$ | $1$ |
| $(x_4, y_3)$ | $0$ |
| $(x_4, y_4)$ | $1$ |
| $(x_5, y_3)$ | $1$ |
| $(x_5, y_4)$ | $2$ |
| $(x_6, y_1)$ | $0$ |
| $(x_6, y_2)$ | $1$ |
| $(x_6, y_5)$ | $3$ |

Figure 4 depicts the generators of $CFK^{\infty}(S^3, K)$; to save space, we do not reproduce this figure here.

As usual, to determine the differentials in $CFK^{\{i=0\}}(S^3, K)$, we destabilize the diagram, as in Figure 17. The situation is more complicated than in the previous two examples, due to the presence of differentials which preserve filtration. These differentials come from domains with $n_z = 0$; there are six such differentials visible in the diagram. There are also many differentials with $n_z = 1$, and these decrease the filtration by 1. Figure 18 summarizes the differentials. All except one are domains of the type discussed in Section 3.3.3. The one tricky differential connects $(x_1, y_3)$ to $(x_3, y_1)$. Its domain contains a tube and has two $270°$ corners. While we can easily compute its Maslov index to be 1, it is less clear how many holomorphic representatives it has. We can deduce, though, that it represents a differential from the requirement that $\partial^2 = 0$. Taking the graded complex $\widehat{CFK}(S^3, K)$ associated to $CFK^{\{i=0\}}(S^3, K)$ and then taking its homology, we get the $E^1$ term of the spectral sequence. This $E^1$ term is $\widehat{HFK}(S^3, K)$; see Figure 19. As before, we can reconstruct $CFK_s^{\infty}(S^3, K)$ from this complex (shown in Figure 20) and use Theorem 3.3 to compute $HF^+(K_{-p}, [m])$ for large $p$.

## 3.6 Proofs of Theorem 3.4 and Theorem 3.3.

### 3.6.1 Proof of Theorem 3.4.

Our first task will be to show that $\Phi$ actually fits into the diagrams in the statement of Theorem 3.4. Indeed, $\Phi$ maps $CFK^{\{i<0 \text{ or } j<0\}}(S^3, K)$ into $CF^-(K_{-p})$, since if $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ represents a component of $\Phi$ from $[\boldsymbol{x}, i, j]$ to $[\boldsymbol{y}, i - n_w(\psi)]$, then $n_w(\psi) - n_z(\psi) = i - j$ by the definition of $\Phi$. Rearranging

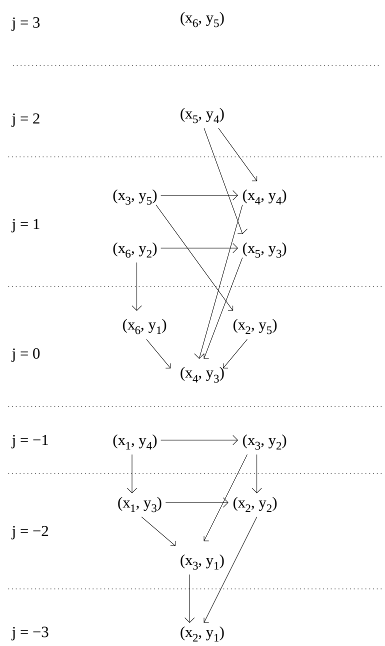


Figure 18: Differentials in the complex $CFK^{\{i=0\}}(S^3, K)$. This complex forms the $E^0$ term of the spectral sequence.

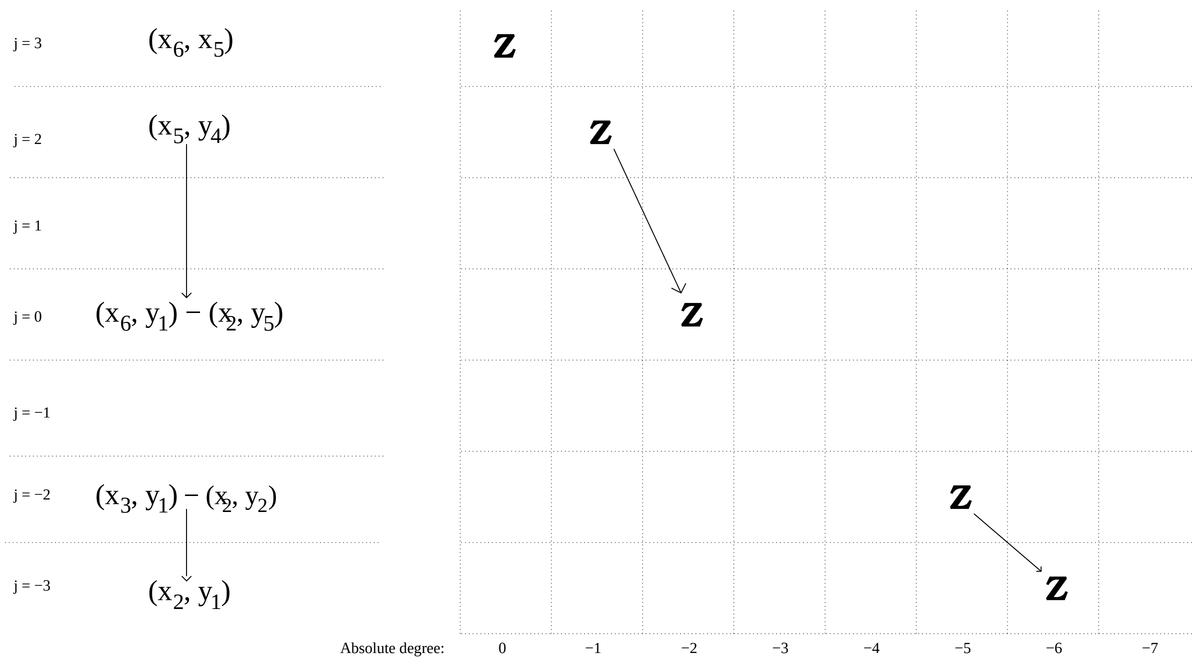


Figure 19: The result of cancelling all filtration-preserving differentials. The element listed at $j = d$ is a generator for $\widehat{HFK}(S^3, K, d)$.

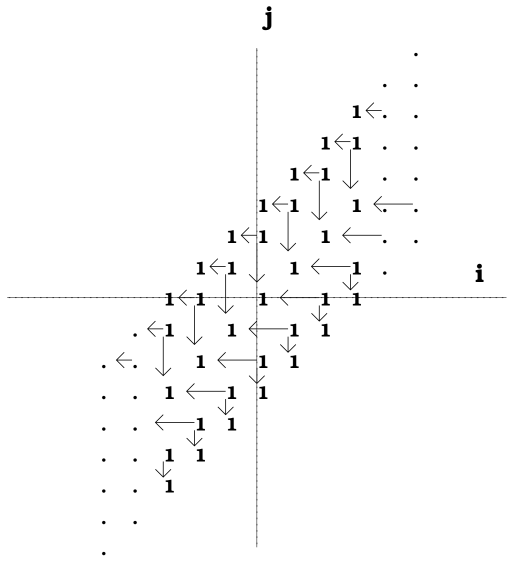

Figure 20: Differentials in $CFK_s^\infty(S^3, K)$.

terms, $i - n_w(\psi) = j - n_z(\psi)$, and since $\psi$ admits a holomorphic representative we know that $n_w(\psi)$ and $n_z(\psi)$ are nonnegative. Hence if $i < 0$ then $i - n_w(\psi) < 0$ and we are done. If $j < 0$, then $j - n_z(\psi) < 0$ too, and so $i - n_w(\psi) = j - n_z(\psi) < 0$. Either way, $[\boldsymbol{x}, i - n_w(\psi)] \in CF^-(K_{-p})$. This computation also shows that $\Phi$ induces a well-defined map on quotient spaces $CFK^{\{i\geq 0, j\geq 0\}}(S^3, K) \xrightarrow{\Phi} CF^+(K_{-p})$.

For the bottom diagram, we need to show that $\Phi$ maps $CFK^{\{\min(i,j)=0\}}(S^3, K)$ into $\widehat{CF}(K_{-p})$. These groups should be interpreted as subgroups of the quotients $CFK^{\{i\geq 0, j\geq 0\}}$ and $CF^+(K_{-p})$, respectively. In other words, if $\min(i,j) = 0$, then we want to have $i - n_w(\psi) \leq 0$ for any $\psi$ contributing to $\Phi$. But if $i = 0$ this follows from the nonnegativity of $n_w(\psi)$, and if $j = 0$ it follows from the nonnegativity of $n_z(\psi)$ together with the requirement $i - n_w(\psi) = j - n_z(\psi)$.

We have shown that $\Phi$ fits into the diagrams without regard for $\mathrm{Spin}^c$ structures; now we want to deal with these. Fix $m$ in $\mathbb{Z}$. It will suffice to show that $\Phi$ maps $CFK^\infty(S^3, K, m)$ into $CF^\infty(K_{-p}, [m])$ for sufficiently large $p$. We need the following lemma:

**Lemma 3.20.** *Suppose $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ for some $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$. Then*

$$\langle c_1(\underline{\mathfrak{s}}(\boldsymbol{x})), [\hat{F}]\rangle + 2(n_w(\psi) - n_z(\psi)) = \langle c_1(\mathfrak{s}_w(\psi)), [S]\rangle - p.$$

*Proof.* See [5], Section 4, p. 80. □

With this lemma in place, let $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ represent a component of $\Phi([\boldsymbol{x}, i, j])$. To say that $[\boldsymbol{y}, i - n_w(\psi)] \in CF^\infty(K_{-p}, [m])$ is to say that $\mathfrak{s}_w(\boldsymbol{y})$ is the restriction of a $\mathrm{Spin}^c$ structure $\mathfrak{r}$ on the surgery cobordism $W_{-p}$ such that $\langle c_1(\mathfrak{r}), [S]\rangle - p \equiv 2m \mod 2p$. But $\mathfrak{s}_w(\boldsymbol{y})$ is the restriction of $\mathfrak{r} = \mathfrak{s}_w(\psi)$, and by Lemma 3.20, we have $\langle c_1(\mathfrak{s}_w(\psi)), [S]\rangle - p = \langle c_1(\underline{\mathfrak{s}}(\boldsymbol{x})), [\hat{F}]\rangle + 2(n_w(\psi) - n_z(\psi))$. The definition of $\Phi$ requires that $n_w(\psi) - n_z(\psi) = i - j$, so

$$\begin{aligned}
\langle c_1(\mathfrak{s}_w(\psi)), [S]\rangle - p &= \langle c_1(\underline{\mathfrak{s}}(\boldsymbol{x})), [\hat{F}]\rangle + 2(i-j) \\
&= \langle c_1(\underline{\mathfrak{s}}(\boldsymbol{x}) + (i-j)PD[\mu]), [\hat{F}]\rangle \\
&= \langle c_1(\mathfrak{t}_m), [\hat{F}]\rangle \\
&= 2m,
\end{aligned}$$

as desired. At this point we have shown that $\Phi$ fits into the diagrams in the statement of Theorem 3.4.

It remains to show that $\Phi$ is an isomorphism between all groups in question. Our strategy will be to decompose $\Phi$ as $\Phi_0+$ lower order terms with respect to the area filtration, where $\Phi_0$ is an isomorphism. Then we will argue that $\Phi$ is an isomorphism since $\Phi_0$ is.

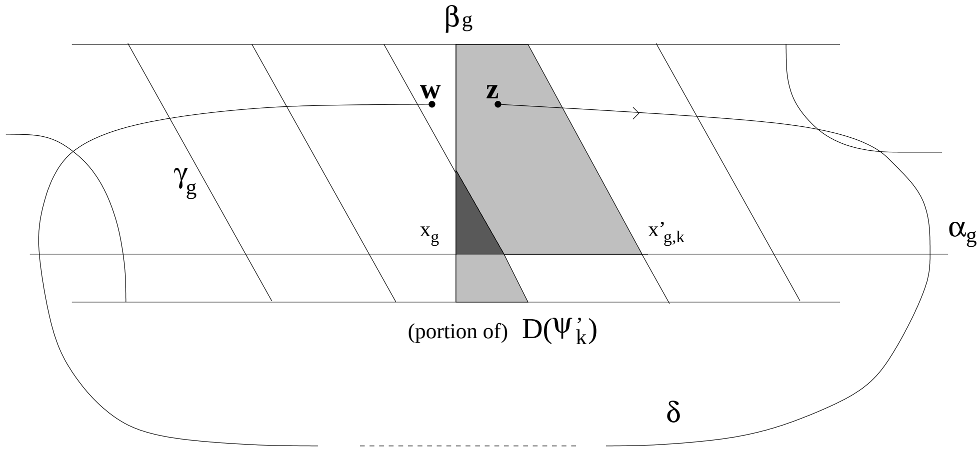


Figure 21: The winding region. The top horizontal line is glued to the bottom horizontal line.

Let $\boldsymbol{x} = \{x_1, \dots, x_g\} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Consider a small neighborhood of $\beta_g$ in $\Sigma$, and assume that the winding of $\gamma_g$ around $\beta_g$ occurs in this region. It will be called the winding region; see Figure 21. Near $x_g$ are $p$ intersection points $\{x'_{g,1}, \dots, x'_{g,p}\}$ of $\alpha_g$ with $\gamma_g$, and near $x_i$ is a unique closest intersection point $x'_i$ of $\alpha_i$ with $\gamma_i$ for $1 \leq i \leq g-1$. In this way, there are $p$ elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$ naturally associated to $\boldsymbol{x}$, labelled $\boldsymbol{x}'_1, \dots, \boldsymbol{x}'_p$ where $\boldsymbol{x}'_k = \{x'_1, \dots, x'_{g-1}, x'_{g,k}\}$. For each $k$, furthermore, there exists a small triangle $\psi_k \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{x}'_k)$. The interesting part of the domain of $\psi_k$ is illustrated in Figure 21. By the Riemann mapping theorem, all of the $\psi_k$ satisfy $\mu(\psi_k) = 0$ and $\#\mathcal{M}(\psi_k) = \pm 1$. Let $[\boldsymbol{x}, i, j]$ be a generator of $CFK^\infty(S^3, K, m)$. For large enough $p$, we claim that there exists a unique $k$ such that $\psi_k$ represents a component of $\Phi$, i.e. such that $n_w(\psi_k) - n_z(\psi_k) = i - j$. This fact can be seen most easily from Figure 21. By examining the domain of $\psi_k$, it is clear that as $k$ runs from 1 to $p$, $n_w(\psi_k) - n_z(\psi_k)$ takes all possible values in a range of about $-p/2$ to $p/2$, exactly once per value. Hence it is equal to $i - j$ exactly once (for large $p$), so there is a unique $\psi_k$ appearing in the expression for $\Phi([\boldsymbol{x}, i, j])$. Let $\psi' = \psi_k$ and $\boldsymbol{x}' = \boldsymbol{x}'_k$ for this unique value of $k$. Now define $\Phi_0([\boldsymbol{x}, i, j]) := [\boldsymbol{x}', i - n_w(\psi')]$.

To show $\Phi_0$ is injective, it suffices to show that we can uniquely reconstruct $[\boldsymbol{x}, i, j]$ from $[\boldsymbol{x}', i - n_w(\psi)]$. But $\boldsymbol{x}$ is the unique point of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ closest to $\boldsymbol{x}'$, and once $\boldsymbol{x}$ is identified, there is a unique small triangle $\psi' \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{x}')$. Then $n_w(\psi')$ is well-defined, and so we can pin down $i$ because we know $i - n_w(\psi)$. Finally, $j$ is uniquely characterized by the property $\underline{\mathfrak{s}}(\boldsymbol{x}) + (i - j)PD[\mu] = \mathfrak{t}_m$.

To show $\Phi_0$ is surjective is a little more subtle. We want to show that any $[\boldsymbol{y}, i] \in CF^\infty(K_{-p}, [m])$ is in the image of $\Phi_0$, where $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$. Write $\boldsymbol{y} = \{y_1, \dots, y_g\}$. The argument of the above paragraph would suffice if we knew $\boldsymbol{y}$ was one of the points $\boldsymbol{x}'_k$ for some $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. This, in turn, would be true if we knew $y_g$ was $x'_{g,k}$ for some intersection point $x_g \in \alpha_g \cap \beta_g$.

We claim that any $\boldsymbol{y}$ representing an element of $CF^\infty(K_{-p}, [m])$ satisfies this condition. Indeed, $[\boldsymbol{y}, i] \in CF^\infty(K_{-p}, [m])$ if and only if $\mathfrak{s}_w(\boldsymbol{y})$ extends over $W_{-p}$ to a Spin$^c$ structure $\mathfrak{r}$ satisfying $\langle c_1(\mathfrak{r}), [S]\rangle - p \equiv 2m \mod 2p$. By Section 2.6.3, this claim amounts to saying that, for any $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, there exists $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}}, \boldsymbol{y})$ with $\langle c_1(s_w(\psi)), [S]\rangle - p \equiv 2m \mod 2p$. But Lemma 3.20 identifies the left-hand side of this equation as $\langle \underline{\mathfrak{s}}(\boldsymbol{x}), [\hat{F}]\rangle + 2(n_w(\psi) - n_z(\psi))$. Now note that the quantities $\langle \underline{\mathfrak{s}}(\boldsymbol{x}), [\hat{F}]\rangle$ and $2m$ do not depend on $p$. Thus, we may assume they are very small compared to $p$. Hence the equation $\langle \underline{\mathfrak{s}}(\boldsymbol{x}), [\hat{F}]\rangle + 2(n_w(\psi) - n_z(\psi)) \equiv 2m \mod 2p$ says that $2(n_w(\psi) - n_z(\psi))$ is very small modulo $2p$ (i.e. is very close to an even multiple of $p$, and is not close to $p \mod 2p$).

But, assuming $y_g$ is not one of the points $x_{g,k}$ for some $x_g \in \alpha_g \cap \beta_g$, we can estimate $n_w(\psi) - n_z(\psi)$ as follows. Choose a path $\delta$ from $z$ to $w$ which does not intersect $\beta_g$. Then, as in Section 2.2.5, we may compute $n_w(\psi) - n_z(\psi)$ by counting the number of times $\delta$ crosses $\partial(D(\psi))$, counted with signs and multiplicity. Now, to obtain $\partial(D(\psi))$, we can start with any 1-chain successively travelling between points of $\boldsymbol{x}$, $\Theta_{\boldsymbol{\beta\gamma}}$, and $\boldsymbol{y}$ along segments of the appropriate $\alpha$, $\beta$, and $\gamma$ curves, and then correct by adding entire copies of $\alpha$, $\beta$, and $\gamma$ curves. In particular, we may start at $x_g$, move to the nearby point of $\Theta_{\boldsymbol{\beta\gamma}}$, and then travel away from $\beta_g$ along $\gamma_g$. When we leave $\gamma_g$ onto an $\alpha$ curve, we will be far from the winding region by assumption, and we will never traverse $\gamma_g$ again.

Hence our 1-chain has multiplicity 1 on about half (about $p/2$) of the "winds" of $\gamma_g$ and multiplicity 0 on the other half, so $\partial(D(\psi))$ has multiplicity $n+1$ on half and multiplicity $n$ on the other half for some $n$. For large $p$, the intersections of $\delta$ with $\partial(D(\psi))$ are approximately equal to the intersections of $\delta$ with the $\gamma_g$-component of $\partial(D(\psi))$, since the number of intersections with other components does not depend on $p$. The intersections of $\delta$ with the $\gamma_g$-component of $\partial(D(\psi))$ add up to about $(2n+1)(p/2)$ for some $n$. Therefore, $2(n_w(\psi) - n_z(\psi)) \sim (2n+1)p \equiv p \mod 2p$. But this was precisely what we said could not hold if $[\boldsymbol{y}, i]$ represented an element of $CF^\infty(K_{-p}, [m])$.

In summary, $CFK^\infty(S^3, K, m) \xrightarrow{\Phi_0} CF^\infty(K_{-p}, [m])$ is an isomorphism. To show it is an isomorphism from $CFK^{\{i<0 \text{ or } j<0\}}(S^3, K, m)$ to $CF^-(K_{-p}, [m])$, we want to show that if $\Phi_0([\boldsymbol{x}, i, j]) = [\boldsymbol{x}', i - n_w(\psi')]$ satisfies $i - n_w(\psi') < 0$, then we must have $i < 0$ or $j < 0$. But if $i - n_w(\psi') < 0$, then $j - n_z(\psi') < 0$ as well, and examining the possible domains of $\psi'$ in Figure 21, we realize that they all satisfy either $n_w(\psi') = 0$ or $n_z(\psi') = 0$. Hence $CFK^{\{i\geq 0, j\geq 0\}}(S^3, K, m) \xrightarrow{\Phi} CF^+(K_{-p}, [m])$ is an isomorphism too, and the same argument shows that $CFK^{\{\min(i,j)=0\}}(S^3, K, m) \xrightarrow{\Phi} \widehat{CF}(K_{-p}, [m])$ is an isomorphism.

Now consider the area filtration on $CF^\infty(K_{-p}, [m])$. If $[\boldsymbol{x}, i, j]$ is a generator of $CFK^\infty(S^3, K, m)$, then for any triangle $\psi$ contributing to $\Phi([\boldsymbol{x}, i, j])$ other than $\psi'$, inspection of Figure 21 shows that the domain of $\psi$ cannot be supported entirely in the winding region. Thus, the corresponding element $[\boldsymbol{y}, i - n_w(\psi)]$ must be strictly lower in filtration degree than $[\boldsymbol{x}', i - n_w(\psi')]$, and we have

$$\Phi = \Phi_0 + \text{ lower order terms.}$$

Furthermore, the area filtration on $CF^\infty(K_{-p}, [m])$ induces a filtration on $CFK^\infty(S^3, K, m)$ via the isomorphism $\Phi_0$. Consider $\Phi_0^{-1}\Phi$; this map may be written as $\mathrm{id} + N$ where $N$ strictly lowers the filtration on $CFK^\infty(S^3, K, m)$. The sum $\sum_{k=0}^\infty (-1)^k N^{\circ k}$ would formally be an inverse for $\Phi_0^{-1}\Phi$ if it were well-defined. Unfortunately, the area filtration on $CF^\infty(K_{-p}, [m])$ is not bounded below, so it is not true that $N^{\circ k} = 0$ for large enough $k$. One can fix this issue by redefining $CF^\infty$ to allow power series in $U$ rather than just polynomials (while still restricting to polynomials in $U^{-1}$). Alternatively, one can cut off $CF^\infty$ at some very low homological degree and then take a limit. Either way, we conclude that $\Phi$ has a left inverse. A similar argument shows $\Phi$ has a right inverse, so it is an isomorphism. One can check that this argument works for all versions of $\Phi$, not just the $\infty$ version (in fact, the version for $CF^+$ is more straightforward since the filtrations are honestly lower-bounded).

### 3.6.2 Proof of Theorem 3.3.

This proof uses the integer surgeries exact triangle, which is discussed in Section 4.3 below. We prove the statement without homological gradings first; then we derive the graded version.

Suppose $|m| > g$. The adjunction inequality (Theorem 2.41) tells us that $HF^+(K_0, m) = 0$. In fact, if $p > g$, then $HF^+(K_0, m') = 0$ for all $m' \cong m \mod p$. The integer surgeries triangle now has the form

$$\begin{tikzcd} & HF^+(S^3) \arrow[dl] & \\ 0 \arrow[rr] & & HF^+(K_{-p}, [m]). \arrow[ul] \end{tikzcd}$$

Hence the map $HF^+(K_{-p}, [m]) \to HF^+(S^3)$ on the right leg of the triangle is an isomorphism.

Now suppose $|m| \leq g$. Theorem 3.4 gives us $HF^+(K_{-p}, [m]) \simeq H_*(CFK^{\{i\geq 0, j\geq 0\}}(S^3, K, m))$. But there is an obvious isomorphism $CFK^{\{i\geq 0, j\geq 0\}}(S^3, K, m) \xrightarrow{\simeq} CFK^{\{i\geq 0, j\geq -m\}}(S^3, K)$ sending $[\boldsymbol{x}, i, j]$ to $[\boldsymbol{x}, i, j-m]$. Thus $HF^+(K_{-p}, [m]) \simeq H_*(CFK^{\{i\geq 0, j\geq -m\}}(S^3, K))$.

It remains to fix the absolute gradings. First, suppose $|m| > g$, and let $W$ denote the surgery cobordism from $K_{-p}$ to $S^3$ discussed in Section 2.6.5. As we will see in Section 4.3.2, the right leg of the triangle above is $\Phi_W$. It is an isomorphism $\mathbb{Z}[U^{-1}] \to \mathbb{Z}[U^{-1}]$, since $HF^+(S^3) = \mathbb{Z}[U^{-1}]$. Hence it may be written as $k_0 + k_1 U^{-1} + k_2 U^{-2} + \cdots$. The terms with nonzero powers of $U$ all decrease homological degree, so by the algebraic argument at the end of the proof of Theorem 3.4, the highest-order term $k_0$ must be an isomorphism. Hence $k_0 = \pm 1$, and we may use this highest-order term (rather than the full $\Phi_W$) as our isomorphism between $HF^+(K_{-p}, [m])$ and $HF^+(S^3)$.

By Proposition 2.64, the highest-order term of $\Phi_W$ comes from Spin$^c$ structures $\mathfrak{r}$ on $W$ which restrict to $[m]$ on $K_{-p}$ and which have the least-negative value of $(c_1(\mathfrak{r}))^2$. In the notation of Section 2.6.5, we consider the $\mathrm{Spin}^c$ structures $\mathfrak{r}_{m'}$ on $W$, where $m' \equiv m \mod p$. Proposition 2.69 tells us that $c_1(\mathfrak{r}_{m'})^2 = -1/p(2m'+p)^2$. For large $p$, this value is maximized when $m' = m$. Thus, the highest-order term shifts degree by $\frac{-1/p(2m+p)^2+1}{4} = \frac{-(2m+p)^2+p}{4p}$. Using this highest-order term as our isomorphism, we get the degree shift claimed in Theorem 3.3.

In the case $|m| \leq g$, note that the isomorphism from $CFK^{\{i\geq 0, j\geq 0\}}(S^3, K, m)$ to $CF^+(K_{-p}, [m])$ is given by the map $\Phi$. The triangles giving rise to components of $\Phi$ are a subset of the triangles giving rise to components of the map $CF^+(S^3) \to CF^+(K_{-p}, [m])$ induced by the surgery cobordism $W_{-p} : S^3 \to K_{-p}$. A triangle $\psi$ shifts absolute degree by $\frac{c_1(\mathfrak{s}_w(\psi))^2 - 2\chi(W) - 3\sigma(W)}{4} = \frac{c_1(\mathfrak{s}_w(\psi))^2+1}{4}$. But by Lemma 3.20, we have $\langle c_1(\mathfrak{s}_w(\psi)), [S]\rangle - p = 2m$ for any triangle in $\Phi$. Hence $\mathfrak{s}_w(\psi) = \mathfrak{r}_m$ for these $\psi$, so Proposition 2.69 tells us that $(c_1(\mathfrak{s}_w(\psi)))^2 = -\frac{1}{p}(2m+p)^2$. Therefore, our isomorphism shifts degrees by $\frac{-(1/p)(2m+p)^2+1}{4} = \frac{-(2m+p)^2+p}{4p}$, as claimed.

# 4 Knots are determined by their complements.

In this section, we will apply a result known as the exact triangle in Heegaard Floer homology, together with a theorem that $\widehat{HFK}$ determines the genus of a knot, to prove the following theorem of Gordon and Luecke [2]:

**Theorem 4.1.** *Let $K_1$ and $K_2$ be two knots in $S^3$. Let $V_i = S^3 \setminus \mathrm{nb}\, K$, for $i = 1, 2$. Suppose that $V_1$ is homeomorphic to $V_2$ via a homeomorphism $\phi$. Then $\phi$ may be extended to an equivalence of knots between $K_1$ and $K_2$, i.e. $\phi$ may be extended to a homeomorphism of $S^3$.*

## 4.1 Overview.

Here we briefly outline our path to Theorem 4.1 before diving into the technical details. Theorem 4.1 will follow as an easy consequence of the following theorem:

**Theorem 4.2.** *Let $K$ be a knot in $S^3$ and let $r$ be a rational number. Suppose the Dehn surgery $K_r$ is homeomorphic to $S^3$. Then $K$ is the unknot.*

This theorem, in turn, follows from the following special case:

**Theorem 4.3.** *Let $K$ be a knot in $S^3$. Suppose the 1-surgery $K_1$ is homeomorphic to $S^3$. Then $K$ is the unknot.*

The equivalence of Theorem 4.2 and Theorem 4.3 will use the "cyclic surgery theorem" of Culler, Gordon, Luecke, and Shalen [1]:

**Theorem 4.4.** *Let $K$ be a knot in $S^3$ and $r = p/q \in \mathbb{Q}$ such that $\pi_1(K_r)$ is cyclic. Either $K$ is the unknot, or $q = \pm 1$.*

We will use this result without proof; see [1]. All of these reductions will be discussed in Section 4.2.

To prove Theorem 4.3, we need an important tool, namely the surgery exact triangle. We state the exact triangle in Section 4.3; we also state a generalized form, the integer surgeries triangle. In Section 4.4, we prove the most basic form of the triangle.

Once the exact triangle is established, we proceed to prove Theorem 4.3. We first show, in Section 4.5, that $HF^+(K_0, i) = 0$ for all $i \neq 0$. In Section 4.6, we prove that if $d$ is the greatest integer for which $\widehat{HFK}(S^3, K, d) \neq 0$, then $\widehat{HFK}(S^3, K, d) \simeq HF^+(K_0, d-1)$, as long as $d > 1$. Putting these two results together, we conclude that $\widehat{HFK}(S^3, K, i) = 0$ for all $i \geq 2$. We then use (without proof) the fact that knot Floer homology detects the genus of a knot. More precisely:

**Theorem 4.5.** *Let $d$ be the greatest integer for which $\widehat{HFK}(S^3, K, d) \neq 0$. Then $d = g$, the Seifert genus of $K$.*

From this result and what has been done so far, we can conclude that $g \leq 1$. Since the unknot is the unique knot with genus 0, we will be done if we can show $g = 0$.

So far, though, we have not excluded the possibility of $g = 1$. To do so, we revisit the arguments of Section 4.5 and Section 4.6 with twisted coefficients. We define Heegaard Floer homology with twisted coefficients in Section 4.7. We then state the relevant generalization of the surgery exact triangle in Section 4.7.2. In Section 4.8, we show that $\underline{HF^+}(K_0) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[t^{\pm 1}, (t-1)^{-1}] = 0$. We then use a result stating that if $d = 1$ is the greatest integer for which $\widehat{HFK}(S^3, K, d) \neq 0$, then $\widehat{HFK}(S^3, K, 1) \otimes \mathbb{Z}[t^{\pm 1}] \simeq \underline{HF}^+(K_0, 0)$. Finally, we cite Theorem 4.5 again to conclude that $g$ must be 0, proving Theorem 4.3 and hence Theorem 4.1.

## 4.2 Reduction of Theorem 4.1 to Theorem 4.3.

### 4.2.1 Reduction to Theorem 4.2.

We begin by showing why Theorem 4.2 implies Theorem 4.1.

*Proof.* (Theorem 4.2 $\Rightarrow$ Theorem 4.1). Suppose we have a homeomorphism $\phi : V_1 \to V_2$. Choose homemorphisms $\psi_i : D^2 \times S^1 \simeq \mathrm{nb}\, K_i$. Then, for both values of $i$, $\psi_i|_{S^1 \times S^1}$ is a homeomorphism from $S^1 \times S^1$ to $\partial(\mathrm{nb}\, K_i) = \partial V_i$, and we can write $S^3 = (D^2 \times S^1) \cup_{\psi_i|_{S^1 \times S^1}} V_i$.

Now consider $\phi \circ \psi_1|_{S^1 \times S^1}$. This gives a homeomorphism from $D^2 \times S^1$ to $\partial(\mathrm{nb}\, K_2) = \partial V_2$. Hence $\phi$ extends to a homeomorphism

$$S^3 = (D^2 \times S^1) \cup_{\psi_1} V_1 \xrightarrow{\phi} (D^2 \times S^1) \cup_{\phi \circ \psi_1} V_2.$$

Let $\mu_i \subset \partial K_i$ be a meridian for $K_i$. The manifold $(D^2 \times S^1) \cup_{\phi \circ \psi_1} V_2$ is obtained from $S^3$ by removing $\mathrm{nb}\, K_2$ and gluing in a solid torus along the curve $\phi(\mu_1)$. If this curve happens to be homologous to $\mu_2$, then $(D^2 \times S^1) \cup_{\phi \circ \psi_1} V_2 \simeq (D^2 \times S^1) \cup_{\psi_2} V_2 = S^3$, and we are done.

Suppose, however, that $\phi(\mu_1)$ is not homologous to $\mu_2$. Then $(D^2 \times S^1) \cup_{\phi \circ \psi_1} V_2$ is obtained from $S^3$ by Dehn surgery on $K_2$; in other words, $(D^2 \times S^1) \cup_{\phi \circ \psi_1} V_2 = (K_2)_r$ for some $r \in \mathbb{Q}$. But we know from the above equation that this surgery must be homeomorphic to $S^3$. By Theorem 4.2, this situation cannot happen unless $K_2$ is the unknot.

If $K_2$ does happen to be the unknot, then repeat the above argument with the roles of $K_1$ and $K_2$ reversed. Either we get the desired extension of $\phi$ this time around, or $K_1$ is also the unknot. In both cases, we are done. □

### 4.2.2 Reduction to Theorem 4.3.

Consider a knot $K$ in $S^3$ such that $K_r \simeq S^3$ for some rational number $r = \frac{p}{q}$. Let $U$ denote the unknot. We begin by dealing with the case $r < 0$:

**Lemma 4.6.** *If Theorem 4.2 holds for all $r \geq 0$, then it holds for all rational $r$.*

*Proof.* Suppose $r < 0$. We have $S^3 \simeq K_r = -(\overline{K}_{-r})$. Thus, reversing orientations, $\overline{K}_{-r} \simeq -S^3$. But $-S^3 \simeq S^3$, so $\overline{K}_{-r} \simeq S^3$. Now $-r > 0$, so by hypothesis we have $\overline{K} \simeq U$. Thus, $K \simeq \overline{U} = U$ as desired. □

Therefore, we can assume that $r = \frac{p}{q}$ with $p$ and $q$ both nonnegative. We next reduce to the case $p = 1$:

**Lemma 4.7.** *If $K_r \simeq S^3$, where $r = \frac{p}{q}$, then $p = 1$.*

*Proof.* A simple argument with the Mayer-Vietoris sequence shows that $H_1(K_r) = \mathbb{Z}/p$. Since $H_1(S^3) = 0$, we must have $p = 1$. □

Finally, we make use of the cyclic surgery theorem, Theorem 4.4, to conclude that $q = 1$ too: we have $K_{1/q} = S^3$, and $\pi_1(S^3)$ is trivial (and hence cyclic) so the conditions of the theorem apply.

## 4.3 Statement of the surgery exact triangle.

We now state the surgery exact triangle and the integer surgeries triangle.

#### 4.3.1 The surgery exact triangle.

Let $K$ be an oriented knot in $S^3$ as usual. There is an exact sequence relating the Heegaard Floer homology of $S^3$, $K_0$, and $K_1$. Note that the zero-surgery $K_0$ has positive first Betti number, so its Heegaard Floer homology is not even relatively $\mathbb{Z}$-graded, and $K_1$ is not absolutely $\mathbb{Z}$-graded. Thus, as in Theorem 2.35, we cannot hope for a traditional homology long exact sequence with boundary maps which lower degree by one. Rather, we will have to settle for an exact triangle in which only three groups appear.

**Theorem 4.8.** *There is an exact triangle*

$$\begin{tikzcd} & \widehat{HF}(S^3) \arrow[dl] & \\ \widehat{HF}(K_0) \arrow[rr] & & \widehat{HF}(K_1). \arrow[ul] \end{tikzcd}$$

*The same holds with $\widehat{HF}$ replaced by $HF^+$.*

In ordinary homology, the maps in the long exact sequence (except for the connecting maps) are induced by continuous maps between the corresponding spaces. Since Heegaard Floer homology is functorial over cobordisms rather than over continuous maps, one might expect the maps in Theorem 4.8 to be induced by cobordisms. This is indeed the case; in fact, all three maps are induced by cobordisms.

#### 4.3.2 Maps in the triangle.

The surgery cobordism from $S^3$ to $K_0$ induces maps $\widehat{HF}(S^3) \to \widehat{HF}(K_0)$ and $HF^+(S^3) \to HF^+(K_0)$, and these are the maps appearing in the triangles above. Furthermore, as we discuss presently, the same applies to each "leg" of the triangle. We can view $K_1$ as surgery on a knot in $K_0$ and $S^3$ as surgery on a knot in $K_1$, and we get associated surgery cobordisms $K_0 \to K_1$ and $K_1 \to S^3$. The induced maps on Heegaard Floer homology coincide with those in the exact triangles.

To obtain $K_1$ as a surgery on a knot in $K_0$, let $K'$ be the knot in $K_0 = (S^3 \setminus \mathrm{nb}\, K) \cup_{\partial(\mathrm{nb}\, K)} (D^2 \times S^1)$ given by $\{0\} \times S^1$. We can take $\mathrm{nb}\, K'$ to be $D^2 \times S^1$, so $K_0 \setminus \mathrm{nb}\, K' = S^3 \setminus \mathrm{nb}\, K$. Now, $K_1$ is obtained by taking $S^3 \setminus \mathrm{nb}\, K$ and gluing in $D^2 \times S^1$ according to the framing given by the curve $\lambda + \mu$, where $\lambda$ is the Seifert longitude for $K$ and $\mu$ is the meridian for $K$. But $\lambda + \mu$ is a curve in $\partial(\mathrm{nb}\, K')$ which intersects the meridian $\lambda$ for $K'$ once. Thus, $\lambda + \mu$ is a longitude for $K'$, and $K_1$ is precisely the manifold obtained from $(\lambda + \mu)$-framed surgery on $K'$.

Similarly, we can obtain $S^3$ as a surgery on the knot $K'' = \{0\} \times S^1$ in $K_1 = (S^3 \setminus \mathrm{nb}\, K) \cup_{\partial(\mathrm{nb}\, K)} (D^2 \times S^1)$. Again, $D^2 \times S^1$ is a tubular neighborhood of $K''$, and removing this neighborhood gives $S^3 \setminus \mathrm{nb}\, K$. Attaching $D^2 \times S^1$ with framing $\mu$ gives $S^3$; but $\mu$ is a longitude for $K''$, so $S^3$ is obtained from $\mu$-framed surgery on $K''$. This construction was already discussed in Section 2.6.5.

Just as we can encode the data of a surgery in a Heegaard triple, we may encode the data of these three surgeries in a Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta})$. We start with a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ describing $K$. As above, we let $\gamma_g = \lambda$ and we take $\gamma_i$ to be a small isotopic translate of $\beta_i$ for $1 \le i \le g$. Also, we take $\delta_g = \lambda + \mu$, and we let $\delta_i$ be a small isotopic translate of $\beta_i$ for $1 \le i \le g$. Then $Y_{\boldsymbol{\alpha\beta}} = S^3$, $Y_{\boldsymbol{\alpha\gamma}} = K_0$, and $Y_{\boldsymbol{\alpha\delta}} = K_1$. We will also require a few admissibility properties for this quadruple; see Proposition 4.10.

The map in the triangle of most interest to us is the one from $\widehat{HF}(K_1)$ to $\widehat{HF}(S^3)$. In the proof of Theorem 4.8 presented below, we will actually get this map as the connecting map in the long exact sequence associated to a short exact sequence of complexes, not explicitly as a cobordism-induced map. However, there are proofs of the exact triangle in which all three maps come explicitly from the natural surgery cobordisms; see [10].

#### 4.3.3 The integer surgeries triangle.

There is a variant of the exact triangle which involves $n$-surgery rather than 1-surgery, which we state here.

**Theorem 4.9.** *Let $[m] \in \mathbb{Z}/n$. There is an exact triangle*

$$\begin{tikzcd} & \widehat{HF}(S^3) \arrow[dl] & \\ \bigoplus_{m' \equiv m \mod n} \widehat{HF}(K_0, m') \arrow[rr] & & \widehat{HF}(K_n, [m]). \arrow[ul] \end{tikzcd}$$

*The same holds with $\widehat{HF}$ replaced by $HF^+$.*

The discussion of surgery cobordisms in Section 4.3.2 applies equally well here; the maps in the triangles are all induced by surgery cobordisms.

## 4.4 Proof of the surgery exact triangle.

Here we present a proof of Theorem 4.8, following Section 9 of [6]. We prove the triangle for $\widehat{HF}$. The result for $HF^+$ takes a bit more work; we focus on $\widehat{HF}$ both for simplicity and because the proof for $HF^+$ is presented very well in [6].

### 4.4.1 The Heegaard quadruple.

We begin by asserting the existence of a Heegaard diagram encoding the data of $Y$, $K_0$, and $K_1$ and satisfying some additional conditions which will be important:

**Proposition 4.10.** *There exists a pointed Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, z)$ and a volume form on $\Sigma$ such that:*

*(a) $Y_{\boldsymbol{\alpha}\boldsymbol{\beta}} = S^3$, $Y_{\boldsymbol{\alpha}\boldsymbol{\gamma}} = K_0$, and $Y_{\boldsymbol{\alpha}\boldsymbol{\delta}} = K_1$;*

*(b) For $i = 1, \dots, g-1$, the $\beta_i$, $\gamma_i$, and $\delta_i$ are small isotopic translates of one another, and they intersect pairwise in two points with opposite sign;*

*(c) Every periodic domain has both positive and negative multiplicities, and has zero area with respect to the volume form; and*

*(d) $\gamma_g$ is isotopic to the concatenation of $\beta_g$ and $\delta_g$.*

The proof involves a bit of Morse theory; see [6].

We will consider a one-parameter family of isotopies of these curves, as follows: for $1 \le i \le g-1$, and for $t \ge 0$, choose curves $\gamma_i(t)$ such that $\gamma_i(0) = \gamma_i$ and $\gamma_i(t) \to \beta_i$ as $t \to \infty$. Choose $\delta_i(t)$ similarly, always preserving the intersection properties of Proposition 4.10. Finally, choose $\gamma_g(t)$ such that $\gamma_g(0) = \gamma_g$ and $\gamma_g(t) \to \beta_g * \delta_g$ as $t \to \infty$, where $*$ denotes the concatenation of paths. We can arrange that at each stage, all periodic domains still have zero area as well as both positive and negative multiplicities. We may also assume that the basepoint $z$ is not in the support of any of these isotopies; thus, at each time $t$, the quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t), z)$ still satisfies the conditions of Proposition 4.10.

For large enough $t$, the curve $\gamma_g(t)$ is very close to the concatenation of $\beta_g$ and $\delta_g$, and $\gamma_i(t)$ is very close to $\beta_i$ for $1 \le i \le g-1$. If $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, let $x_i \in \boldsymbol{x}$ be the point of intersection between $\beta_i$ and some $\alpha_j$. Since $\gamma_i(t)$ approaches very close to $\beta_i$, there is a unique closest intersection point $x_i'$ between $\alpha_j$ and $\gamma_i(t)$. These points form the element $\boldsymbol{x'} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$. We have produced a mapping $\iota : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$ and hence a homomorphism $I : \widehat{CF}(S^3) \to \widehat{CF}(K_0)$. The same argument produces a map $\rho : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)} \to \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$ and thus a homomorphism $R : \widehat{CF}(K_1) \to \widehat{CF}(K_0)$; note that for $1 \le i \le g-1$, the curve $\delta_i(t)$ approaches $\beta_i$, and hence it also approaches $\gamma_i(t)$.

If $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$, let $y_g$ be the intersection point in $\boldsymbol{y}$ between $\gamma_g(t)$ and some $\alpha_i$. If $t$ is large enough, $y_g$ is very close to either an intersection between $\alpha_i$ and $\beta_g$ or an intersection between $\alpha_i$ and $\delta_g$. Hence $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$ splits up into the image of $\iota$ and the image of $\rho$. As an Abelian group, there is a corresponding splitting of $\widehat{CF}(K_0)$ as $\widehat{CF}(S^3) \oplus \widehat{CF}(K_1)$, where the inclusions of these two groups into $\widehat{CF}(K_0)$ are given by $I$ and $R$. Let $L$ and $P$ be the corresponding projections of $\widehat{CF}(K_0)$ onto $\widehat{CF}(S^3)$ and $\widehat{CF}(K_1)$ respectively. We have a split short exact sequence of groups (although not of chain complexes):

$$0 \to \widehat{CF}(S^3) \xrightarrow{I} \widehat{CF}(K_0) \xrightarrow{P} \widehat{CF}(K_1) \to 0.$$

To prove Theorem 4.8, we will obtain an honest short exact sequence of chain complexes. The sequence above will be the highest-order part with respect to area filtrations we will define on each complex.

For use in Section 4.4.3, we will make further requirements which can be satisfied by choosing $t$ large enough. For each $i$ between 1 and $g-1$, there is a doubly periodic domain $P_i$ formed between $\beta_i$ and $\gamma_i(t)$. There is another doubly periodic domain $Q_i$ formed between $\beta_i$ and $\delta_i(t)$. Finally, there is a triply periodic domain $P$ formed between $\beta_g$, $\gamma_g(t)$, and $\delta_g$. The unsigned area of each of these domains approaches 0 as $t$ increases. Define

$$\epsilon(t) = \sum_i \mathcal{A}(|P_i|) + \mathcal{A}(|Q_i|),$$

where the absolute value signs denote the unsigned area. Then $\epsilon(t) \to 0$ as $t \to \infty$. We may also assume that, for each $t$, we have $\mathcal{A}(|Q_i|) = \mathcal{A}(|P_i|)$, so that bounding $\epsilon$ also bounds the size of the periodic domains $Q_i$. Let $M$ be the minimum area of any component of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g \cup \delta_g\}$. Now choose $t$ large enough that $\epsilon(t)$ is very small compared to $M$. Choosing $\epsilon(t) < M/4$ should suffice.

### 4.4.2 Proof of the exact triangle given appropriate filtrations.

Recall that we had maps $\widehat{CF}(S^3) \xrightarrow{f_1} \widehat{CF}(K_0)$ and $\widehat{CF}(K_0) \xrightarrow{f_2} \widehat{CF}(K_1)$ defined by counting holomorphic triangles in the Heegaard triples $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}(t), z)$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t), z)$. These maps were the chain maps induced by the surgery cobordisms $S^3 \to K_0$ and $K_0 \to K_1$. We want the maps in our triangle to be the maps $F_1$ and $F_2$ induced by $f_1$ and $f_2$ on homology. For convenience, we recall concretely how $f_1$ and $f_2$ work: for $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\boldsymbol{x}' \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$, we have

$$f_1(\boldsymbol{x}) = \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}, \psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta}\boldsymbol{\gamma}(t)}, \boldsymbol{y}) | \mu(\psi)=0, n_z(\psi)=0\}} \#\mathcal{M}(\psi) \cdot \boldsymbol{y}$$

and

$$f_2(\boldsymbol{x}') = \sum_{\{\boldsymbol{y}' \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}, \psi \in \pi_2(\boldsymbol{x}', \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}), \boldsymbol{y}' | \mu(\psi)=0, n_z(\psi)=0\}} \#\mathcal{M}(\psi) \cdot \boldsymbol{y}'.$$

As is often the case when constructing a long exact homology sequence, we will obtain our exact triangle from a short exact sequence of chain complexes. Since $f_1$ and $f_2$ are chain maps, the first possibility one might consider would be $0 \to \widehat{CF}(S^3) \xrightarrow{f_1} \widehat{CF}(K_0) \xrightarrow{f_2} \widehat{CF}(K_1) \to 0$. However, this sequence is not exact. Using filtrations and the maps from Section 4.4.1, we will modify $f_1$ by a chain homotopy to obtain exactness. More precisely, we have the following theorem:

**Theorem 4.11.** *There exists a chain map $g_1 : \widehat{CF}(S^3) \to \widehat{CF}(K_0)$, chain homotopic to $f_1$, such that*

$$0 \to \widehat{CF}(S^3) \xrightarrow{g_1} \widehat{CF}(K_0) \xrightarrow{f_2} \widehat{CF}(K_1) \to 0$$

*is a short exact sequence of chain complexes.*

This result implies Theorem 4.8. Furthermore, the maps $\widehat{HF}(S^3) \to \widehat{HF}(K_0)$ and $\widehat{HF}(K_0) \to \widehat{HF}(K_1)$ are $F_1$ and $F_2$ (since $g_1$ also induces $F_1$ on homology), which are the cobordism-induced maps. As mentioned earlier, this proof gets the map $\widehat{HF}(K_1) \to \widehat{HF}(S^3)$ as a connecting homomorphism, but one can show using other methods that this map is also induced by the natural surgery cobordism. We will devote the rest of our efforts in Section 4.4.2 and Section 4.4.3 to proving Theorem 4.11.

For the rest of Section 4.4.2, we will assume the following lemma asserting the existence of appropriate filtrations on the complexes in question:

**Lemma 4.12.** *There exist filtrations on $\widehat{CF}(S^3)$, $\widehat{CF}(K_0)$, and $\widehat{CF}(K_1)$ satisfying the following properties:*

*(a) The filtrations are bounded below.*

*(b) The boundary maps in the complexes are strictly filtration-decreasing.*

*(c) The maps $I$ and $R$ are filtration-preserving.*

*(d) We have $f_1 = I+$ lower order terms; in other words, for $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $f_1(\boldsymbol{x}) - I(\boldsymbol{x}) < I(\boldsymbol{x})$ in the filtration on $\widehat{CF}(K_0)$.*

*(e) We have $f_2|_{\operatorname{im} R} = P+$ lower order terms; in other words, for $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$, $f_2(R\boldsymbol{y}) - \boldsymbol{y} < \boldsymbol{y}$ in the filtration on $\widehat{CF}(K_1)$.*

*(f) The composition $f_2 \circ f_1$ is chain homotopic to zero via a chain homotopy $H : \widehat{CF}(S^3) \to \widehat{CF}(K_1)$ which is filtration-decreasing, meaning that for $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $RH\boldsymbol{x} < I\boldsymbol{x}$ in the filtration on $\widehat{CF}(K_0)$.*

The lemma will be proved in Section 4.4.3, with the exception of the existence of $H$. To prove that part would require a discussion of coherent orientation systems for moduli spaces, and we have avoided mention of these.

Given Lemma 4.12, we can prove Theorem 4.11. We begin by defining a right inverse for $f_2$. By Lemma 4.12, we can write $f_2 \circ R = \mathrm{id} + N$, where $N$ is strictly filtration-decreasing. Let

$$R' := R \circ \sum_{k=0}^{\infty} (-1)^k N^{\circ k}.$$

Because the filtration on $\widehat{CF}(K_1)$ is bounded below, sufficiently high powers of $N$ are zero, so the sum is finite and hence well-defined. We have $f_2 \circ R' = (\mathrm{id} + N) \circ \sum_{k=0}^{\infty} (-1)^k N^{\circ k} = \mathrm{id}$, so $R'$ is a right inverse for $f_2$. This already shows $f_2$ is surjective. $R'$ is a sum of powers of a filtration-decreasing map, so it is filtration non-increasing.

We can also use $R'$ to define $g_1$: let

$$g_1 := f_1 - (\partial R' H + R' H \partial).$$

Then $g_1$ is chain homotopic to $f_1$ via the chain homotopy $R'H$. Furthermore, since $\partial$ and $H$ decrease filtration and $R'$ does not increase it, while $f_1 = I+$ lower order terms by Lemma 4.12, we have $g_1 = I+$ lower order terms. Say $g_1 = I + E$, where $E$ satisfies $E < I$. Postcomposing with $L$, we get $L \circ g_1 = \mathrm{id} + (L \circ E)$. Because $I$ preserves filtration, $E$ decreases it, and we know $L$ does not increase it. Hence we may write $g_1 = \mathrm{id} + N'$ where $N'$ strictly decreases the filtration on $\widehat{CF}(S^3)$. Now define

$$L' = \sum_{k=0}^{\infty} (-1)^k (N')^{\circ k} \circ L.$$

Again, the sum defining $L'$ is finite, and $L' \circ g_1 = (\sum_{k=0}^{\infty} (-1)^k (N')^{\circ k})(\mathrm{id} + N') = \mathrm{id}$. Hence $L'$ is a left inverse for $g_1$, so $g_1$ is injective.

To check exactness in the middle term, first note that by Lemma 4.12,

$$\begin{aligned} f_2 \circ g_1 &= f_2 \circ f_1 - f_2(\partial R' H + R' H \partial) \\ &= f_2 \circ f_1 - (\partial (f_2 R') H + (f_2 R') H \partial) \\ &= f_2 \circ f_1 - (\partial H + H \partial) \\ &= 0. \end{aligned}$$

So we know $\operatorname{im} g_1 \subset \ker f_2$; in other words, we know our sequence is a chain complex. We want to show its homology is zero. It is a filtered chain complex because we have filtrations on all groups in question and the "differentials" $g_1$ and $f_2$ are filtration non-increasing. Hence we can compute the homology of the complex via a spectral sequence. The $E_1$ term is obtained by replacing the maps in the complex by their filtration-preserving parts and taking homology. This procedure amounts to replacing $g_1$ by $I$ and $f_2$ by $P$, obtaining the sequence of Section 4.4.1, and taking homology. But this sequence is exact. Hence the $E_1$ term of the sequence is already 0, so the homology of the complex we started with must also be 0. Thus $\operatorname{im} g_1 = \ker f_2$, and we are done.

### 4.4.3 Construction of filtrations with the right properties.

In this section, we prove Lemma 4.12. We are given a Heegaard quadruple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t), z)$, with $Y_{\boldsymbol{\alpha\beta}} = S^3$, $Y_{\boldsymbol{\alpha\gamma}(t)} = K_0$, and $Y_{\boldsymbol{\alpha\delta}(t)} = K_1$. Thus we have area filtrations $\mathcal{F}^{area}_{S^3}$, $\mathcal{F}^{area}_{K_0}$, and $\mathcal{F}^{area}_{K_1}$ on $\widehat{CF}(S^3)$, $\widehat{CF}(K_0)$, and $\widehat{CF}(K_1)$ defined by areas of disks, triangles, and squares as in Section 2.6.6. These, unfortunately, do not quite give us the filtrations we want on $\widehat{CF}(S^3)$, $\widehat{CF}(K_0)$, and $\widehat{CF}(K_1)$, since the maps $I$ and $R$ do not preserve them. However, they are almost the correct filtrations. We will start with the area filtration $\mathcal{F}_{K_0} := \mathcal{F}^{area}_{K_0}$ on $\widehat{CF}(K_0)$ from triangles and use $I$ and $R$ to transport it to $\widehat{CF}(S^3)$ and $\widehat{CF}(K_1)$. We will then show that the resulting filtrations $\mathcal{F}_{S^3}$ and $\mathcal{F}_{K_1}$ on $\widehat{CF}(S^3)$ and $\widehat{CF}(K_1)$ differ from $\mathcal{F}^{area}_{S^3}$ and $\mathcal{F}^{area}_{K_1}$ by no more than $\epsilon(t)$. This will imply that $\mathcal{F}_{S^3}$ and $\mathcal{F}_{K_1}$ are actually filtrations, i.e. that the differential decreases them.

We now fill in the details. For a generator $\boldsymbol{x}$ of $\widehat{CF}(S^3)$, define $\mathcal{F}_{S^3}(\boldsymbol{x}) := \mathcal{F}_{K_0}(I(\boldsymbol{x}))$. For a generator $\boldsymbol{y}$ of $\widehat{CF}(K_1)$, define $\mathcal{F}_{K_1}(\boldsymbol{y}) := \mathcal{F}_{K_0}(R(\boldsymbol{y}))$. Then, by decree, $I$ and $R$ preserve filtration. All three filtrations are bounded below because they are defined on finitely-generated chain complexes. The following proposition shows that the differential is strictly filtration-decreasing:

**Proposition 4.13.** *The differential strictly decreases all three filtrations $\mathcal{F}_{S^3}$, $\mathcal{F}_{K_0}$, and $\mathcal{F}_{K_1}$.*

*Proof.* For $\mathcal{F}_{K_0}$, the result follows from Section 2.6.6.

For $\mathcal{F}_{S^3}$, suppose that $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. To compute $\mathcal{F}^{area}_{S^3}(\boldsymbol{x})$, we can choose a disk $\phi \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$; then $\mathcal{F}^{area}_{S^3}(\boldsymbol{x}) = -\mathcal{A}(D(\phi))$. But $\boldsymbol{x}$ is very close to $\iota(\boldsymbol{x}) \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$, and so there is a small triangle $\psi_0 \in \pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \iota(\boldsymbol{x}))$. Since the support of $D(\psi_0)$ is contained inside the total support of the periodic domains, we have $|\mathcal{A}(D(\psi_0))| < \epsilon(t)$. Now the concatenation $\phi * \psi_0$ is a triangle in $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \iota(\boldsymbol{x}))$, so it may be used to compute $\mathcal{F}_{K_0}(\iota(\boldsymbol{x})) =: \mathcal{F}_{S^3}(\boldsymbol{x})$. We thus have $\mathcal{F}_{S^3}(\boldsymbol{x}) = -\mathcal{A}(D(\phi * \psi_0)) = -\mathcal{A}(D(\phi)) - \mathcal{A}(D(\psi_0)) = \mathcal{F}^{area}_{S^3}(\boldsymbol{x}) - \mathcal{A}(D(\psi_0))$. Hence

$$|\mathcal{F}_{S^3}(\boldsymbol{x}) - \mathcal{F}^{area}_{S^3}(\boldsymbol{x})| \leq |\mathcal{A}(D(\psi_0))| < \epsilon(t).$$

This inequality holds for all $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$.

Now suppose $\boldsymbol{x}$ and $\boldsymbol{x}'$ are two elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, and suppose $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{x}')$ represents a differential. Then the difference in $\mathcal{F}^{area}_{S^3}$ between $\boldsymbol{x}$ and $\boldsymbol{x}'$ is at least $\mathcal{A}(D(\phi))$. But $D(\phi)$ is a positive sum of components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g\}$, and each one of these components has area at least $M$. Hence the difference in $\mathcal{F}^{area}_{S^3}$ between $\boldsymbol{x}$ and $\boldsymbol{x}'$ is at least $M$. But $\mathcal{F}^{area}_{S^3}(\boldsymbol{x})$ and $\mathcal{F}^{area}_{S^3}(\boldsymbol{x}')$ differ from $\mathcal{F}_{S^3}(\boldsymbol{x})$ and $\mathcal{F}_{S^3}(\boldsymbol{x}')$ by at most $\epsilon(t)$, and so $\mathcal{F}_{S^3}(\boldsymbol{x}) - \mathcal{F}_{S^3}(\boldsymbol{x}')$ is at least $M - 2\epsilon(t)$. Since $\epsilon(t)$ is very small compared to $M$, we see that the differential decreases $\mathcal{F}_{S^3}$.

Next we consider $\mathcal{F}_{K_1}$. Suppose $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$. Then $\rho(\boldsymbol{y}) \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$, and we can choose a triangle $\psi \in \pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \rho(\boldsymbol{y}))$. We have $\mathcal{F}_{K_1}(\boldsymbol{y}) = \mathcal{F}_{K_0}(\rho(\boldsymbol{y})) = -\mathcal{A}(D(\psi))$. There is also a small triangle $\psi_0$ in $\pi_2(\rho(\boldsymbol{y}), \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y})$. The support of $D(\psi_0)$ is contained in the support of the periodic domains, so $|\mathcal{A}(D(\psi_0))| < \epsilon(t)$. But we can concatenate the triangles $\psi$ and $\psi_0$ to obtain a "pinched" square $\psi * \psi_0 \in \pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y})$. Such squares compute $\mathcal{F}^{area}_{K_1}$, so we have $\mathcal{F}^{area}_{K_1}(\boldsymbol{y}) = -\mathcal{A}(D(\psi * \psi_0)) = -\mathcal{A}(D(\psi)) - \mathcal{A}(D(\psi_0)) = \mathcal{F}_{K_1}(\boldsymbol{y}) - \mathcal{A}(D(\psi_0))$. Hence

$$|\mathcal{F}_{K_1}(\boldsymbol{y}) - \mathcal{F}^{area}_{K_1}(\boldsymbol{y})| \leq |\mathcal{A}(D(\psi_0))| < \epsilon(t).$$

This inequality holds for all $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$.

Suppose $\boldsymbol{y}$ and $\boldsymbol{y}'$ are in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$ and $\phi \in \pi_2(\boldsymbol{y}, \boldsymbol{y}')$ represents a differential. The difference in $\mathcal{F}^{area}_{K_1}$ between $\boldsymbol{y}$ and $\boldsymbol{y}'$ is at least $\mathcal{A}(D(\phi))$. But $D(\phi)$ is a positive sum of components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \delta_1(t) \cup \cdots \cup \delta_{g-1}(t) \cup \delta_g\}$. Furthermore, since each $\delta_i(t)$ is very close to $\beta_i$ for $1 \leq i \leq g-1$, we may make a small (i.e. supported in the periodic domains and hence of size $\leq \epsilon(t)$) adjustment to $D(\phi)$ to obtain a positive sum of components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_{g-1} \cup \delta_g\}$. This modified domain has area at least $M$, so we can conclude that $\mathcal{A}(D(\phi)) \geq M - \epsilon(t)$. Hence the difference in $\mathcal{F}^{area}_{K_1}$ between $\boldsymbol{y}$ and $\boldsymbol{y}'$ is at least $M - \epsilon(t)$. But, again, $\mathcal{F}^{area}_{K_1}$ differs from $\mathcal{F}_{K_1}$ by at most $\epsilon(t)$, so we can conclude that $\mathcal{F}_{K_1}(\boldsymbol{y}) - \mathcal{F}_{K_1}(\boldsymbol{y}')$ is at least $M - 3\epsilon(t)$. Since $\epsilon(t)$ is small compared to $M$, we see that the differential decreases $\mathcal{F}_{K_1}$.

□

Next we want decompositions $f_1 = I+$ lower order and $f_2|_{\operatorname{im} R} = P+$ lower order.

**Proposition 4.14.** *For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, we have $f_1(\boldsymbol{x}) = I(\boldsymbol{x})+$ lower order terms, and for $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$, we have $f_2(R(\boldsymbol{y})) = \boldsymbol{y}+$ lower order terms.*

*Proof.* Consider the small triangle $\psi_0 \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}(t)}, \iota(\boldsymbol{x}))$. Using a characterization of Maslov indices of triangles similar to that found in Section 3.3.2 for disks, one can show that $\mu(\psi_0) = 0$. The usual argument with the Riemann mapping theorem then shows that $\#\mathcal{M}(\psi_0) = \pm 1$, and if we were careful with orientation systems, we could deduce $\#\mathcal{M}(\psi_0) = 1$. Hence one component of $f_1(\boldsymbol{x})$ is $\iota(\boldsymbol{x}) = I(\boldsymbol{x})$.

We would like to show that any other $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}(t)}$ appearing in the expression for $f_1(\boldsymbol{x})$ must be of lower order than $\iota(\boldsymbol{x})$. Suppose that, for some $\boldsymbol{y} \neq \iota(\boldsymbol{x})$, there exists $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}(t)}, \boldsymbol{y})$ representing a component of $f_1(\boldsymbol{x})$. A quick glance at the periodic domains shows that only $D(\psi_0)$ can be entirely supported in them; thus, $D(\psi)$ cannot be supported inside the periodic domains. We know $D(\psi)$ is a positive sum of components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g \cup \gamma_1(t) \cup \cdots \cup \gamma_g(t)\}$. However, since $\gamma_i(t)$ is very close to $\beta_i$ for $1 \leq i \leq g-1$, we may assume (after a modification of size $< \epsilon(t)$) that $D(\psi)$ is a positive sum of components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g \cup \gamma_g(t)\}$. Since $\gamma_g(t)$ is very close to the concatenation of $\beta_g$ and $\delta_g$, and since $D(\psi)$ is not supported entirely in the narrow region between $\gamma_g(t)$, $\beta_g$, and $\delta_g$, we may further assume (with another small error term $< \epsilon(t)$) that $D(\psi)$ is a positive sum of components of $\Sigma \setminus \{\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g \cup \delta_g\}$. Hence, up to a correction of $2\epsilon(t)$, we have $\mathcal{A}(D(\psi)) \geq M$, or in other words $\mathcal{A}(D(\psi)) \geq M - 2\epsilon(t)$.

Choose $\phi \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$ with $n_z(\phi) = 0$; then the concatenation $\phi * \psi$ computes $\mathcal{F}_{K_0}(\boldsymbol{y})$, so we have

$$\begin{aligned} \mathcal{F}_{K_0}(\boldsymbol{y}) &= -\mathcal{A}(D(\phi)) - \mathcal{A}(D(\psi)) \\ &\leq -A(D(\phi)) - M + 2\epsilon(t). \end{aligned}$$

But we also have $-\mathcal{A}(D(\phi)) = \mathcal{F}^{area}_{S^3}(\boldsymbol{x})$, and we saw in the proof of Proposition 4.13 that $\mathcal{F}^{area}_{S^3}(\boldsymbol{x})$ is nearly equal to $\mathcal{F}_{S^3}(\boldsymbol{x}) = \mathcal{F}_{K_0}(\iota(\boldsymbol{x}))$. In particular, $\mathcal{F}^{area}_{S^3}(\boldsymbol{x}) \leq \mathcal{F}_{K_0}(\iota(\boldsymbol{x})) + \epsilon(t)$, and so $\mathcal{F}_{K_0}(\boldsymbol{y}) \leq \mathcal{F}_{K_0}(\iota(\boldsymbol{x})) - M + 3\epsilon(t)$. Rearranging terms,

$$\mathcal{F}_{K_0}(\boldsymbol{y}) - \mathcal{F}_{K_0}(\iota(\boldsymbol{x})) \leq -M + 3\epsilon(t) < 0,$$

and we have verified that $f_1 = I+$ lower order terms.

Now we deal with $f_2$. If $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}}$, consider $f_2(R(\boldsymbol{y}))$. As above, there is a small triangle in $\pi_2(\rho(\boldsymbol{y}), \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y})$ giving rise to a component of $f_2(R(\boldsymbol{y}))$ on $\boldsymbol{y}$ with coefficient $+1$. Suppose $\boldsymbol{w} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$ is any other point appearing in the expression for $f_2(R(\boldsymbol{y}))$, via some triangle $\psi \in \pi_2(\rho(\boldsymbol{y}), \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{w})$. We want to show that $\boldsymbol{w}$ is lower in the filtration that $\boldsymbol{y}$, or equivalently (since $R$ preserves filtration) that $\rho(\boldsymbol{w})$ is lower than $\rho(\boldsymbol{y})$ in the filtration on $\widehat{CF}(K_0)$.

First, consider $D(\psi)$. By arguments analogous to those above, we may conclude that $A(D(\psi)) \geq M - 2\epsilon(t)$. Now, to compute the filtration gradings of $\rho(\boldsymbol{y})$ and $\rho(\boldsymbol{w})$, pick triangles $\psi_{\boldsymbol{y}}$ in $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \rho(\boldsymbol{y}))$ and $\psi_{\boldsymbol{w}}$ in $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \rho(\boldsymbol{w}))$ satisfying $n_z(\psi_{\boldsymbol{y}}) = n_z(\psi_{\boldsymbol{w}}) = 0$. Then $\mathcal{F}_{K_0}(\rho(\boldsymbol{y})) = -\mathcal{A}(D(\psi_{\boldsymbol{y}}))$ and $\mathcal{F}_{K_0}(\rho(\boldsymbol{w})) = -\mathcal{A}(D(\psi_{\boldsymbol{w}}))$. The concatenation $\psi_{\boldsymbol{y}} * \psi$ is a square in $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{w})$. But if $\psi_0$ denotes the small triangle in $\pi_2(\rho(\boldsymbol{w}), \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{w})$, then the concatenation $\psi_{\boldsymbol{w}} * \psi_0$ is another square in $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta\gamma}(t)}, \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{w})$. Since all periodic domains have zero area, both $\psi_{\boldsymbol{y}} * \psi$ and $\psi_{\boldsymbol{w}} * \psi_0$ must have the same area. We can conclude that $\mathcal{A}(D(\psi_{\boldsymbol{y}})) + \mathcal{A}(D(\psi)) = \mathcal{A}(D(\psi_{\boldsymbol{w}})) + \mathcal{A}(D(\psi_0))$, or (rearranging and relabelling) that $\mathcal{F}_{K_0}(\rho(\boldsymbol{w})) - \mathcal{F}_{K_0}(\rho(\boldsymbol{y})) = \mathcal{A}(D(\psi_0)) - \mathcal{A}(D(\psi))$. But $\mathcal{A}(D(\psi_0)) \leq \epsilon(t)$, while $\mathcal{A}(D(\psi)) \geq M - 2\epsilon(t)$. Thus,

$$\mathcal{F}_{K_0}(\rho(\boldsymbol{w})) - \mathcal{F}_{K_0}(\rho(\boldsymbol{y})) \leq -M + 3\epsilon(t) < 0,$$

and we have shown that $f_2|_{\operatorname{im} R} = P+$ lower order terms as desired. □

We are left with the final assertion of Lemma 4.12. As mentioned above, we will simply cite the proof of Theorem 8.16 of [7] for the existence of $H$. In fact, $H$ is defined by counting holomorphic squares: if $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, then we have

$$H(\boldsymbol{x}) := \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}, \varphi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\gamma}(t)}, \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y}) | \mu(\varphi) = -1, n_z(\varphi) = 0\}} \#\mathcal{M}(\varphi) \cdot \boldsymbol{y}.$$

To apply the construction of [7], one first shows that for suitable orientation systems, $f_2 \circ f_1$ induces the zero map on homology. For details, see Section 9 of [6]. We will simply show that, given this $H$, it decreases filtrations in the sense that $R \circ H < I$.

**Proposition 4.15.** *For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, we have $R(H(\boldsymbol{x})) < I(\boldsymbol{x})$.*

*Proof.* Suppose $H(\boldsymbol{x})$ has nonzero coefficient on some $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\delta}(t)}$ coming from some square $\varphi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta}\boldsymbol{\gamma}(t)}, \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y})$. We want to show that $\rho(\boldsymbol{y}) < \iota(\boldsymbol{y})$ in $\mathcal{F}_{K_0}$. Pick $\psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta},\boldsymbol{\gamma}(t)}, \rho(\boldsymbol{y}))$ with $n_z(\phi) = 0$. There is also a small triangle $\psi_0 \in \pi_2(\rho(\boldsymbol{y}), \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y})$. The concatenation $\psi * \psi_0$ is a square in $\pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta}\boldsymbol{\gamma}(t)}, \Theta_{\boldsymbol{\gamma}(t)\boldsymbol{\delta}(t)}, \boldsymbol{y})$, and $n_z(\psi * \psi_0) = 0$. Thus $\varphi$ and $\psi * \psi_0$ differ by a periodic domain. Since periodic domains have zero area, we see that $\mathcal{A}(D(\varphi)) = \mathcal{A}(D(\psi)) + \mathcal{A}(D(\psi_0))$.

Pick $\phi \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$ with $n_z(\phi) = 0$. Then $\phi * \psi$ is an element of $\pi_2(\boldsymbol{x}_0, \Theta_{\boldsymbol{\beta}\boldsymbol{\gamma}(t)}, \rho(\boldsymbol{y}))$ with $n_z = 0$, so we have $-\mathcal{A}(D(\phi)) - \mathcal{A}(D(\psi)) = \mathcal{F}_{K_0}(\rho(\boldsymbol{y}))$. Furthermore, by the proof of Proposition 4.13, we know that $-\mathcal{A}(D(\phi)) = \mathcal{F}^{area}_{S^3}(\boldsymbol{x})$ is no more than $\epsilon(t)$ away from $\mathcal{F}_{K_0}(\iota(\boldsymbol{x}))$. Hence, up to an error term of $\epsilon(t)$, we have $\mathcal{F}_{K_0}(\iota(\boldsymbol{x})) - \mathcal{F}_{K_0}(\rho(\boldsymbol{y})) = \mathcal{A}(D(\psi))$. More precisely, we have

$$\begin{aligned}\mathcal{F}_{K_0}(\iota(\boldsymbol{x})) - \mathcal{F}_{K_0}(\rho(\boldsymbol{y})) &\geq \mathcal{A}(D(\psi)) - \epsilon(t)\\ &= \mathcal{A}(D(\varphi)) - \mathcal{A}(D(\psi_0)) - \epsilon(t)\\ &\geq \mathcal{A}(D(\varphi)) - 2\epsilon(t).\end{aligned}$$

But inspection shows that $\varphi$ cannot be supported entirely inside the periodic domains. The arguments of Proposition 4.14, applied to the square $\varphi$ rather than to triangles, thus imply that $\mathcal{A}(D(\varphi))$ must be at least $M - 2\epsilon(t)$, so in the end we have

$$\mathcal{F}_{K_0}(\iota(\boldsymbol{x})) - \mathcal{F}_{K_0}(\rho(\boldsymbol{y})) \geq M - 4\epsilon(t) > 0.$$

This inequality proves the proposition and hence Lemma 4.12, Theorem 4.11, and Theorem 4.8. $\square$

## 4.5 The vanishing of $HF^+(K_0, i)$ when $i \neq 0$.

Since $K_1 \simeq S^3$, we have $\widehat{HF}(K_1) \simeq \mathbb{Z}$. Thus, the surgery exact triangle is

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl] & \\ \widehat{HF}(K_0) \arrow[rr] & & \mathbb{Z} \arrow[ul, "\varphi"'] \end{tikzcd}$$

While one could show that $\varphi = 0$, for our purposes this fact is unnecessary. The map $\varphi$ is either zero or it is multiplication by some nonzero integer $m$. We consider both cases.

Suppose $\varphi = 0$. Then $\widehat{HF}(K_0) \simeq \mathbb{Z}^2$. However, $\widehat{HF}(K_0)$ can be written as $\oplus_{i\in\mathbb{Z}}\widehat{HF}(K_0, i)$. Since $b_1(K_0) = 1$, we must have $\chi(\widehat{HF}(K_0, i)) = 0$ for all $i$ by Proposition 2.39. Hence both copies of $\mathbb{Z}$ in $\widehat{HF}(K_0)$ must live in the same Spin$^c$ structure.

Assume $\widehat{HF}(K_0, i) = \mathbb{Z}^2$ and $\widehat{HF}(K_0, j) = 0$ for all $j \neq i$. Proposition 2.38 tells us that $\widehat{HF}(K_0, -i)$ is $\mathbb{Z}^2$ as well, so we must have $i = -i$. In other words $i = 0$, and we have proved $\widehat{HF}(K_0, i) = 0$ for $i \neq 0$ assuming $\varphi = 0$. By Proposition 2.40, $HF^+(K_0, i) = 0$ for $i \neq 0$.

On the other hand, suppose $\varphi$ is multiplication by $m \neq 0$. Then $\varphi$ is injective, so we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \widehat{HF}(K_0) \to 0.$$

Therefore, $\widehat{HF}(K_0) \simeq \mathbb{Z}/m$, a cyclic group. Again, we can conclude that $\widehat{HF}(K_0)$ is supported in only one Spin$^c$ structure and (by conjugation symmetry) that this Spin$^c$ structure must be $i = 0$. Hence $\widehat{HF}(K_0, i) = 0$ for $i \neq 0$. By Proposition 2.40, $HF^+(K_0, i) = 0$ for $i \neq 0$.

## 4.6 $\widehat{HFK}(S^3, K)$ and $HF^+(K_0)$ in the highest nonzero Spin$^c$ structure.

In this section, we will prove the following statement:

**Proposition 4.16.** *Let $d$ be the largest integer such that $\widehat{HFK}(S^3, K, d) \neq 0$, and suppose $d > 1$. (In fact, by Theorem 4.5, $d = g$, the Seifert genus of $K$). Then $\widehat{HFK}(S^3, K, d) \simeq HF^+(K_0, d - 1)$.*

*Proof.* Consider the following short exact sequence of subcomplexes of $CFK^\infty(S^3, K)$:

$$0 \to CFK^{\{i<0, j\geq d-1\}}(S^3, K) \to CFK^{\{i\geq 0 \text{ or } j\geq d-1\}}(S^3, K) \to CFK^{\{i\geq 0\}}(S^3, K) \to 0.$$

As usual, there is an associated long exact sequence in homology, in the form of a triangle since there is no natural absolute $\mathbb{Z}$-grading. To make use of it, we want to identify the homology of these three complexes. We first claim that $H_*(CFK^{\{i<0,j\geq d-1\}}(S^3, K)) = \widehat{HFK}(S^3, K, d)$. Indeed, the filtrations on the complex allow us to compute its homology by cancelling differentials step-by-step. The first step is to consider only differentials which keep both $i$ and $j$ fixed. With these differentials, the complex splits up as $\oplus_{\{i<0,j\geq d-1\}}\widehat{CFK}(S^3, K, i + j)$. Hence the result of cancelling these filtration-preserving differentials is $\oplus_{\{i<0,j\geq d-1\}}\widehat{HFK}(S^3, K, i + j)$. But, by assumption, $\widehat{HFK}(S^3, K, m) = 0$ for $m \geq d$, so this sum is just $\widehat{HFK}(S^3, K, d)$ as claimed.

Next, we make use of Theorem 3.5 to identify the homology of $CFK^{\{i\geq 0 \text{ or } j\geq d-1\}}(S^3, K)$, or equivalently $CFK^{\{i\geq 0 \text{ or } j\geq 0\}}(S^3, K, d-1)$, as $HF^+(K_n, [d-1])$ for a sufficiently large choice of $n$. Finally, to compute $H_*(CFK^{\{i\geq 0\}}(S^3, K))$, we simply note that, ignoring the $j$-filtration, $CFK^{\{i\geq 0\}}(S^3, K) = CF^+(S^3)$. Hence, its homology is $HF^+(S^3)$.

We can compare the exact triangle just derived with the integer surgeries triangle:

$$\begin{tikzcd}[column sep=small] & HF^+(S^3) \arrow[dl] & \\ \widehat{HFK}(S^3, K, d) \arrow[rr] & & HF^+(K_n, [d-1]) \arrow[ul, "\Theta"'] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & HF^+(S^3) \arrow[dl] & \\ HF^+(K_0, d-1) \arrow[rr] & & HF^+(K_n, [d-1]) \arrow[ul, "\Phi_W"'] \end{tikzcd}$$

The map $\Phi_W$ is induced from the cobordism $W$ from $K_1$ to $S^3$. In the integer surgeries triangle, we would usually have $\sum_{m\equiv d-1 \mod n} HF^+(K_0, m)$ in the left-hand corner. However, $n$ is large compared to $d$, so any other integers $m \equiv d-1 \mod n$ are much greater than $d-1$ in absolute value. Since $d$ is the Seifert genus of $K$, the adjunction inequality (Theorem 2.41) tells us that $HF^+(K_0, m) = 0$ for all such $m$. Thus, only one term appears in the exact triangle.

While these triangles suggest a relationship between $\widehat{HFK}(S^3, K, d)$ and $HF^+(K_0, d-1)$, we need more information about the maps in the triangles to derive equality. Specifically, we will look at the maps $\Phi_W$ and $\Theta$. We first note that $\Theta$ is surjective. Indeed, for a sufficiently large choice of absolute degree $N$, the projection of $CFK^{\{i\geq 0 \text{ or } j\geq d-1\}}_{\deg\geq N}(S^3, K)$ onto $CFK^{\{i\geq 0\}}_{\deg\geq N}(S^3, K)$ is an isomorphism, where these two complexes are generated by elements of $CFK^{\{i\geq 0 \text{ or } j\geq d-1\}}(S^3, K)$ and $CFK^{\{i\geq 0\}}(S^3, K)$, respectively, with absolute grading $\geq N$. This statement holds because the complex $CFK^{i=0}(S^3, K)$ has finitely many generators (corresponding to elements of $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$), and the absolute degrees of these generators are thus bounded above, say by $M$. For $N \geq M$, then, any generator of $CFK^{\{i\geq 0 \text{ or } j\geq d-1\}}_{\deg\geq N}(S^3, K)$ which has degree $\geq N$ must actually have $i \geq 0$. Therefore, since $\Theta$ is the map on homology induced by a chain map which is an isomorphism in sufficiently large degrees, $\Theta$ is also an isomorphism in sufficiently large degrees. Writing $HF^+(S^3)$ as $\mathbb{Z}[U^{-1}]$, we see that $U^{-k}$ is in the image of $\Theta$ for all sufficiently large $k$. But since $\Theta$ is $U$-equivariant, this fact actually implies that $U^{-k}$ is in the image of $\Theta$ for all $k \geq 0$. Hence $\Theta$ is surjective.

The surjectivity of $\Phi_W$ is a bit less elementary. We know from Section 4.3.2 that $\Phi_W$ breaks apart as a sum over Spin$^c$ structures on $W$. First we will show that $\Theta$ is the component of $\Phi_W$ associated to one particular Spin$^c$ structure on $W$. Next, we will establish that the components of $\Phi_W$ corresponding to all the other Spin$^c$ structures are of lower order with respect to the absolute grading. An algebraic argument will then show that $\Phi_W$ is surjective, and a further algebraic manipulation will conclude the proof.

To identify $\Theta$ as coming from a Spin$^c$ structure on $W$, consider the following diagram detailing the identifications made above:

$$\begin{tikzcd}
CFK^{\{i\geq 0 \text{ or } j\geq d-1\}}(S^3,K) \arrow[r, "\text{proj}"] \arrow[d, equal] & CFK^{\{i\geq 0\}}(S^3,K) \arrow[d, equal] \\
CFK^{\{i\geq 0 \text{ or } j\geq 0\}}(S^3,K,d-1) & CF^+(S^3) \\
CF^+(K_n,[d-1]) \arrow[u, "\Psi", "\simeq"'] \arrow[ur, "\Phi_{W,\mathfrak{r}}"'] &
\end{tikzcd}$$

Here $\Psi$ is the map from Theorem 3.5, and $\Phi_{W,\mathfrak{r}}$ is the map induced by some Spin$^c$ structure $\mathfrak{r}$ on $W$. Taking the top path around the diagram induces $\Theta$ on homology. We claim that the diagram commutes when $\mathfrak{r}$ is chosen so that $\langle c_1(\mathfrak{r}),[S]\rangle = 2(d-1)+n$, where $[S]$ generates $H_2(W)$. In the notation of Section 2.6.5, $\mathfrak{r}=\mathfrak{r}_{d-1}$.

The claim follows from an analogue of Lemma 3.20. More precisely, suppose $\mathfrak{r}=\mathfrak{s}_w(\psi)$ for some triangle $\psi\in\pi_2(\boldsymbol{x},\Theta_{\boldsymbol{\beta\gamma}},\boldsymbol{y})$, where $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a Heegaard diagram for $K_n$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma})$ is a Heegaard diagram for $S^3$ as in the definition of $\Psi$. Here, $\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$ and $\boldsymbol{y}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\gamma}}$. Then

$$\langle c_1(\mathfrak{r}),[S]\rangle - n = \langle c_1(\mathfrak{s}(\boldsymbol{y})),[\hat{F}]\rangle + 2(n_w(\psi)-n_z(\psi)).$$

The definition of $\Psi$ counts only triangles $\psi$ with $(1/2)\langle c_1(\mathfrak{s}(\boldsymbol{y})),[\hat{F}]\rangle + n_w(\psi)-n_z(\psi)=d-1$, and we see that these are precisely the triangles belonging to the Spin$^c$ structure $\mathfrak{r}$ with $\langle c_1(\mathfrak{r}),[S]\rangle = 2(d-1)+n$. In other words, they are exactly the triangles making up $\Phi_{W,\mathfrak{r}_d}$ as claimed.

Now, the components of $\Phi_W$ other than $\Theta$ will all be induced by Spin$^c$ structures $\mathfrak{r}'$ on $W$ with $\langle c_1(\mathfrak{r}'),[S]\rangle = 2m+n$ for some $m\equiv d-1 \bmod p$. For sufficiently large $n$, all such $m$ will be very large in absolute value. But if $\langle c_1(\mathfrak{r}'),[S]\rangle = 2m+n$, then the map induced by $\mathfrak{r}$ shifts the absolute grading by $\frac{c_1(\mathfrak{r}')^2+1}{4} = \frac{-(2m+n)^2/n+1}{4} = \frac{-(2m+n)^2+n}{4n}$ by Proposition 2.69, and so when $n$ is large the decrease is minimized (with respect to the constraint $m\equiv d-1 \bmod n$) when $m=d-1$.

Write $\Phi_W=\Theta+L$, where $L$ contains the contributions from all the other Spin$^c$ structures. Since $\Theta$ is surjective, we can choose a right inverse $R$ for $\Theta$ (since $HF^+(S^3)$ is a free Abelian group). Define an automorphism $K$ of $CF^+(K_n,[d-1])$ by the formula

$$K=\sum_{k\geq 0}(-1)^k(R\circ L)^{\circ k}.$$

Since $R$ increases the absolute grading by $\frac{(2(d-1)+n)^2-n}{4n}$, while $L$ decreases it by at least $\frac{(2m+n)^2-n}{4n}$ for $m>>d-1$, we see that $R\circ L$ decreases the absolute grading, and so (since the absolute grading is bounded below in $HF^+$) we have $(R\circ L)^{\circ k}=0$ for large enough $k$. Hence the sum in the definition of $R'$ is well-defined. In fact, $K$ is the identity plus a term which lowers absolute degree, and this lower-degree term is nilpotent because the absolute grading is bounded below. Thus $K$ is an automorphism of $CF^+(K_n,[d-1])$. Finally,

$$\begin{aligned}
\Phi_W\circ K &= (\Theta+L)\circ K \\
&= \Theta+L-(\Theta+L)\circ(R\circ L)+(\Theta+L)\circ(R\circ L)^{\circ 2}-\cdots \\
&= \Theta+L-L+L\circ R\circ L-L\circ R\circ L+\cdots \\
&= \Theta.
\end{aligned}$$

The first thing we can conclude from the above computation is that $\Phi_W$ is surjective because $\Theta$ is. Now the exact triangles become short exact sequences:

$$0\to\widehat{HFK}(S^3,K,d)\to HF^+(K_n,[d-1])\overset{\Theta}{\to} HF^+(S^3)\to 0$$

and

$$0\to HF^+(K_0,d-1)\to HF^+(K_n,[d-1])\overset{\Phi_W}{\to} HF^+(S^3)\to 0.$$

Hence $\widehat{HFK}(S^3,K,d)\simeq\ker\Theta$ and $HF^+(K_0,d-1)\simeq\ker\Phi_W$. But the automorphism $K$ above restricts to an isomorphism of $\ker\Theta$ with $\ker\Phi_W$. Therefore, $\widehat{HFK}(S^3,K,d)\simeq HF^+(K_0,d-1)$ as desired. $\square$

## 4.7 Heegaard Floer homology with twisted coefficients.

### 4.7.1 The twisted-coefficient groups.

We discuss here the simplest case of Heegaard Floer homology with twisted coefficients. Suppose $b_1(Y) = 1$. Then $H^1(Y) = \mathbb{Z}$, and the Heegaard Floer homology of $Y$ with twisted coefficients, $\underline{\widehat{HF}}(Y)$, will be a module over the group ring $\mathbb{Z}[H^1(Y)] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t^{\pm 1}]$, the ring of Laurent polynomials. For a discussion of the general case, see Section 8 of [6].

To make the definition, pick a weakly admissible Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for $Y$. Choose a designated point $\boldsymbol{x}_0 \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and a disk $\phi_{\boldsymbol{x}} \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Then, for each $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, we may identify $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ with $\pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$ by mapping $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ to $\phi_{\boldsymbol{y}}^{-1} * \phi * \phi_{\boldsymbol{x}}$ in $\pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$.

The choice of $\phi_{\boldsymbol{x}_0} \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$ allows us to further define a map from $\pi_2(\boldsymbol{x}_0, \boldsymbol{x}_0)$ to $H^1(Y)$. It takes a disk $\phi$ and looks at the difference between its domain $D(\phi)$ and the domain $D(\phi_0)$ of $\phi_0$. This difference is a periodic domain and hence defines an element of $H_2(Y) = H^1(Y)$.

Finally, choose an identification of $H^1(Y)$ with $\mathbb{Z}$. These choices give rise to a mapping $A : \pi_2(\boldsymbol{x}, \boldsymbol{y}) \to \mathbb{Z}$ for any $\boldsymbol{x}, \boldsymbol{y}$, enabling the following definition:

**Definition 4.17.** Suppose $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

(a) As a group, $\underline{\widehat{CF}}(Y, \mathfrak{s})$ is defined to be $\widehat{CF}(Y, \mathfrak{s}) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$.

(b) For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$,

$$\underline{\partial}(\boldsymbol{x} \otimes 1) := \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, \phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) | \mu(\phi)=1, n_z(\phi)=0\}} \# \overline{\mathcal{M}}(\phi) \cdot (\boldsymbol{y} \otimes t^{A(\phi)}).$$

Extending equivariantly in $t$, we get a differential on $\underline{\widehat{CF}}(Y, \mathfrak{s})$. It satisfies $\underline{\partial}^2 = 0$; we define $\underline{\widehat{HF}}(Y, \mathfrak{s}) = \frac{\ker \underline{\partial}}{\mathrm{im}\, \underline{\partial}}$.

We have an analogous definition for $\underline{HF^+}$:

**Definition 4.18.** Suppose $\mathfrak{s} \in \mathrm{Spin}^c(Y)$.

(a) As a group, $\underline{CF^+}(Y, \mathfrak{s})$ is defined to be $CF^+(Y, \mathfrak{s}) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$.

(b) For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$,

$$\underline{\partial}([\boldsymbol{x}, i] \otimes 1) := \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, \phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) | \mu(\phi)=1\}} \# \overline{\mathcal{M}}(\phi) \cdot ([\boldsymbol{y}, i - n_z(\phi)] \otimes t^{A(\phi)}).$$

Extending equivariantly in $t$, we get a differential on $\underline{CF^+}(Y, \mathfrak{s})$. It satisfies $\underline{\partial}^2 = 0$; we define $\underline{HF^+}(Y, \mathfrak{s}) = \frac{\ker \underline{\partial}}{\mathrm{im}\, \underline{\partial}}$.

While we made several choices in the definitions above, the homology groups $\underline{\widehat{HF}}(Y, \mathfrak{s})$ and $\underline{HF^+}(Y, \mathfrak{s})$ are independent of them.

Now suppose $Y$ is the zero-surgery $K_0$ on a knot $K$ in $S^3$. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ be a Heegaard triple consistent with the surgery data as in Section 2.6.2. Note that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, z)$ is a Heegaard diagram for $K_0$. In this case, there is a more concrete way to make the choices described above. Recall that $\gamma_g$ is the Seifert longitude for $K$. Choose a reference point $\tau \in \gamma_g$ disjoint from the $\alpha$ and $\beta$ curves. Let $V$ be the codimension-1 submanifold $\gamma_1 \times \cdots \times \gamma_{g-1} \times \{\tau\} \subset \mathbb{T}_{\boldsymbol{\gamma}}$. Then if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$ and $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, we can define $A(\phi)$ to be the intersection number $\partial_\gamma(\phi) \cdot V$. Here, $\partial_\gamma(\phi)$ stands for $(\partial(\mathrm{im}\, \phi)) \cap \mathbb{T}_{\boldsymbol{\gamma}}$. This choice of $A$ works just as well as the $A$ obtained from making the choices described at the beginning of the section.

### 4.7.2 Statements of the exact triangles with twisted coefficients.

We have versions of the surgery exact triangle and integer surgery triangle using the twisted-coefficient groups:

**Theorem 4.19.** *There is an exact triangle*

$$\begin{tikzcd}
 & \widehat{HF}(S^3) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \arrow[dl] & \\
\underline{\widehat{HF}}(K_0) \arrow[rr] & & \widehat{HF}(K_1) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]. \arrow[ul]
\end{tikzcd}$$

*The same holds with $\widehat{HF}$ replaced by $HF^+$.*

**Theorem 4.20.** *Suppose $[m] \in \mathbb{Z}/n$. There is an exact triangle*

$$\begin{tikzcd}
 & \widehat{HF}(S^3) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}] \arrow[dl] & \\
\bigoplus_{m' \equiv m \mod n} \underline{\widehat{HF}}(K_0, m') \arrow[rr] & & \widehat{HF}(K_n, [m]) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]. \arrow[ul]
\end{tikzcd}$$

*The same holds with $\widehat{HF}$ replaced by $HF^+$.*

As before, the maps are induced by cobordisms. The one which is relevant for us is the map $\widehat{HF}(K_1) \otimes \mathbb{Z}[t^{\pm 1}] \to \widehat{HF}(S^3) \otimes \mathbb{Z}[t^{\pm 1}]$. The corresponding map in the untwisted triangle was induced by the surgery cobordism $W$ from $K_1$ to $S^3$ and split up according to Spin$^c$ structures: $\Phi_W = \sum_{\mathfrak{r} \in \mathrm{Spin}^c(W)} \Phi_{W,\mathfrak{r}}$. Analogously, we will call the map in the twisted triangle $\underline{\Phi}_W$. To state how $\underline{\Phi}_W$ is defined on the chain level, let $\tau'$ be a reference point in $\delta_g$ which is disjoint from the other attaching circles. Let $V' = \delta_1 \times \cdots \times \delta_{g-1} \times \{\tau'\} \subset \mathbb{T}_{\boldsymbol{\delta}}$. The following equation defines $\underline{\Phi}_W$:

$$\underline{\Phi}_W([\boldsymbol{x}, i] \otimes t^k) = \sum_{\{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, \psi \in \pi_2(\boldsymbol{x}, \Theta_{\boldsymbol{\beta\delta}}, \boldsymbol{y}) | \mu(\psi)=0\}} \#(\mathcal{M}(\psi)) \cdot ([\boldsymbol{x}, i - n_z(\psi)] \otimes t^{k + \partial_\delta(\psi) \cdot V'}). \tag{5}$$

As it turns out, there exists some $M \in \mathbb{Z}$ such that Equation 5 simplifies as follows:

$$\underline{\Phi}_W = \sum_{\mathfrak{r} \in \mathrm{Spin}^c(W)} \Phi_{W,\mathfrak{r}} \otimes t^{\frac{\langle c_1(\mathfrak{r}), [S] \rangle + M}{2}}. \tag{6}$$

As usual, $[S]$ is the generator of $H_2(W) = \mathbb{Z}$ defined in Section 2.6.4. Also, as we will see, it does not really matter what the value of $M$ is. Clearly, though, it must have the same parity as $\langle c_1(\mathfrak{r}), [S] \rangle$. In other words, $M$ must be odd since $\langle c_1(\mathfrak{r}), [S] \rangle$ is odd for all $\mathfrak{r} \in \mathrm{Spin}^c(W)$.

### 4.7.3 Proposition 4.16 with twisted coefficients.

We have the following twisted analogue of Proposition 4.16:

**Proposition 4.21.** *Suppose $d = 1$ is the largest integer such that $\widehat{HFK}(S^3, K, d) \neq 0$. Then the group $\widehat{HFK}(S^3, K, 1) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$ is isomorphic to $\underline{HF^+}(K_0, 0)$.*

*Proof.* The proof of Proposition 4.16 carries over word-for-word until we introduce the integer surgeries exact triangle. In the twisted case, we use the twisted version of the integer surgeries triangle. The two triangles we want to relate are

$$\begin{tikzcd}
 & HF^+(S^3) \otimes \mathbb{Z}[t^{\pm 1}] \arrow[dl] & \\
\widehat{HFK}(S^3, K, 1) \otimes \mathbb{Z}[t^{\pm 1}] \arrow[rr] & & HF^+(K_n, [0]) \otimes \mathbb{Z}[t^{\pm 1}] \arrow[ul, "\Theta \otimes \mathrm{id}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & HF^+(S^3)\otimes_{\mathbb{Z}}\mathbb{Z}[t^{\pm 1}] \arrow[dl] & \\
\underline{HF^+}(K_0,0) \arrow[rr] & & HF^+(K_n,[0])\otimes_{\mathbb{Z}}\mathbb{Z}[t^{\pm 1}]. \arrow[ul, "\underline{\Phi}_W"']
\end{tikzcd}$$

The surjectivity of $\Theta\otimes\mathrm{id}$ follows from the surjectivity of $\Theta$ in the proof of Proposition 4.16. For $\Phi_W$, we previously had $\Psi_W = \Theta + L$, where $L$ represents lower-order terms with respect to the area filtration. This formula, together with Equation 6 for $\underline{\Phi}_W$, becomes $\underline{\Phi}_W = \Theta\otimes t^{\frac{n+M}{2}} + L'$. Here $L'$ is still a sum of terms which are lower-order with respect to the area filtration (which, in this context, ignores $t$). To verify this equation, note that if $\mathfrak{r}$ is the Spin$^c$ structure on $W$ giving rise to $\Theta$, then $\langle c_1(\mathfrak{r}),[S]\rangle = 2(d-1)+n = n$ since $d=1$.

Now, if $R$ is a right inverse for $\Theta$ as before, then $R\otimes\mathrm{id}$ is a right inverse for $\Theta\otimes\mathrm{id}$. The same algebraic argument, using $R\otimes t^{-\frac{n+M}{2}}$ in place of $R$ and $L'$ in place of $L$, gives us an automorphism $K' = \sum_{k\geq 0}(-1)^k((R\otimes t^{-\frac{n+M}{2}})\circ L')^{\circ k}$ of $CF^+(K_1,[0])\otimes\mathbb{Z}[t^{\pm 1}]$. We have $\underline{\Phi}_W\circ K = \Theta\otimes t^{\frac{n+M}{2}}$, since

$$\begin{aligned}
\underline{\Phi}_W\circ K' &= (\Theta\otimes t^{\frac{n+M}{2}} + L')\circ K' \\
&= \Theta\otimes t^{\frac{n+M}{2}} + L' - (\Theta\otimes t^{\frac{n+M}{2}} + L')\circ(R\otimes t^{-\frac{n+M}{2}}\circ L') \\
&\quad + (\Theta\otimes t^{\frac{n+M}{2}} + L')\circ(R\otimes t^{-\frac{n+M}{2}}\circ L')^{\circ 2} - \cdots \\
&= \Theta\otimes t^{\frac{n+M}{2}} + L' - L' + L'\circ(R'\otimes t^{-\frac{n+M}{2}})\circ L' - L'\circ(R\otimes t^{-\frac{n+M}{2}})\circ L' + \cdots \\
&= \Theta\otimes t^{\frac{n+M}{2}}.
\end{aligned}$$

But $t^{\frac{n+M}{2}}$ is invertible in $\mathbb{Z}[t^{\pm 1}]$. Hence $\underline{\Phi}_W$ is surjective and its kernel is isomorphic (via $(K')^{-1}\otimes t^{\frac{n+M}{2}}$) to that of $\Theta\otimes\mathrm{id}$, completing the proof.

$\square$

### 4.8 The vanishing of $\underline{HF^+}(K_0)\otimes_{\mathbb{Z}[t^{\pm 1}]}\mathbb{Z}[t^{\pm 1},(t-1)^{-1}]$.

Since $K_1\simeq S^3$, we have $HF^+(K_1) = HF^+(S^3) = \mathbb{Z}[U^{-1}]$. Hence the twisted surgery exact triangle becomes

$$\begin{tikzcd}
 & \mathbb{Z}[U^{-1},t^{\pm 1}] \arrow[dl] & \\
\underline{HF^+}(K_0) \arrow[rr] & & \mathbb{Z}[U^{-1},t^{\pm 1}]. \arrow[ul]
\end{tikzcd}$$

Inverting $t-1$, we get

$$\begin{tikzcd}
 & \mathbb{Z}[U^{-1},t^{\pm 1},(t-1)^{-1}] \arrow[dl] & \\
\underline{HF^+}(K_0)\otimes_{\mathbb{Z}[t^{\pm 1}]}\mathbb{Z}[t^{\pm 1},(t-1)^{-1}] \arrow[rr] & & \mathbb{Z}[U^{-1},t^{\pm 1},(t-1)^{-1}]. \arrow[ul, "\underline{\Phi}_W"']
\end{tikzcd}$$

Here, $W$ is the surgery cobordism from $K_1$ to $S^3$. We will be done if we can show that the right leg $\underline{\Phi}_W$ of the triangle is an isomorphism. Our plan of attack will be to show that, in terms of the absolute $\mathbb{Q}$-grading, $\underline{\Phi}_W$ is an isomorphism plus a lower-order term. Using algebra, we will then be able to show that $\underline{\Phi}_W$ is itself an isomorphism.

As in Section 2.6.5, let $\mathfrak{r}_m$ be the Spin$^c$ structure on $W$ with $\langle c_1(\mathfrak{r}_m),[S]\rangle - n = 2m$. By Proposition 2.69, the map $\Phi_{W,\mathfrak{r}_m}$ shifts degree by $\frac{-(2m+1)^2+1}{4} = -m^2-m$. This shift is always $\leq 0$, and it equals zero precisely when $m=0$ or $m=-1$. Thus, only the Spin$^c$ structures $\mathfrak{r}_0$ and $\mathfrak{r}_{-1}$, with first Chern classes $[S]^*$ and $-[S]^*$, contribute to the highest-degree term of $\underline{\Phi}_W$.

Using Equation 6, we may write the highest-degree term of $\underline{\Phi}_W$ as $\Phi_{W,\mathfrak{r}_0}\otimes t^{\frac{1+M}{2}} + \Phi_{W,\mathfrak{r}_{-1}}\otimes t^{\frac{-1+M}{2}}$. Both $\Phi_{W,\mathfrak{r}_0}$ and $\Phi_{W,\mathfrak{r}_{-1}}$ are maps from $\mathbb{Z}[U^{-1}]$ to $\mathbb{Z}[U^{-1}]$ which do not decrease absolute degrees. Thus, they are multiplication by integers $c_0$ and $c_{-1}$ respectively. The highest-degree term of $\underline{\Phi}_W$ is therefore $c_0 t^{\frac{1+M}{2}} + c_{-1}t^{\frac{-1+M}{2}}$.

**Lemma 4.22.** *Both $c_0$ and $c_{-1}$ are $\pm 1$.*

*Proof.* By Proposition 9.4 of [4], both $\Phi_{W,\mathfrak{r}_0}$ and $\Phi_{W,\mathfrak{r}_{-1}}$ are isomorphisms on $HF^\infty$. Their domain and codomain are both $\mathbb{Z}[U, U^{-1}]$, so both maps are multiplication by $\pm U^k$ for some $k$. But the generator of $HF^\infty(K_1)$ has the same absolute degree as the generator of $HF^\infty(S^3)$, namely 0, because $K_1$ is equal to $S^3$. Thus, since $\Phi_{W,\mathfrak{r}_0}$ and $\Phi_{W,\mathfrak{r}_{-1}}$ preserve absolute degrees, they must both be multiplication by $\pm 1$.

To relate this result to the maps on $HF^+$, consider the exact sequence of Theorem 2.35. Usually, we only have an exact triangle. However, $S^3$ has a Heegaard diagram with no differentials. Thus, we can derive this sequence from a short exact sequence of complexes $0 \to CF^-(S^3) \to CF^\infty(S^3, \to CF^+(S^3) \to 0$ in which taking homology changes nothing. Theorem 2.35 becomes a short exact sequence, and Theorem 2.66 gives us a diagram

$$\begin{tikzcd}
0 \arrow[r] & HF^-(S^3) \arrow[r] \arrow[d] & HF^\infty(S^3) \arrow[r] \arrow[d, "\Phi_{W,\mathfrak{r}_0}=\pm 1"] & HF^+(S^3) \arrow[r] \arrow[d, "\Phi_{W,\mathfrak{r}_0}"] & 0 \\
0 \arrow[r] & HF^-(S^3) \arrow[r] & HF^\infty(S^3) \arrow[r] & HF^+(S^3) \arrow[r] & 0.
\end{tikzcd}$$

Since every element of $HF^+(S^3)$ comes from $HF^\infty(S^3)$, we see that $\Phi_{W,\mathfrak{r}_0}$ is multiplication by $\pm 1$ on $HF^+(S^3)$. The same holds for $\Phi_{W,\mathfrak{r}_{-1}}$. $\square$

Hence the highest-degree term of $\underline{\Phi}_W$ is multiplication by either $\pm t^{M'}(t+1)$ or $\pm t^{M'}(t-1)$ for some integer $M'$. Actually, $\underline{\Phi}_W$ must be 0 when $t$ is set equal to 1, so the second option is correct. This fact is not necessary for our purposes, though; if the first option were correct instead, we could simply have inverted $t+1$ instead of $t-1$. In any case, the highest-degree term of $\underline{\Phi}_W$ is an isomorphism. An algebraic argument like the one in the proof of Theorem 3.4, using the absolute $\mathbb{Q}$-grading rather than the area filtration, implies that $\underline{\Phi}_W$ is itself an isomorphism. Therefore $\underline{HF^+}(K_0) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Z}[t^{\pm 1}, (t-1)^{-1}] = 0$, completing the proof.

## 4.9 Conclusion of the proof of Theorem 4.3.

The unknot is the unique knot with genus 0, so it will suffice to prove that the genus $g$ of $K$ is 0. By Theorem 4.5, we know that $g$ is the largest value of $d$ for which $\widehat{HFK}(S^3, K, d) \neq 0$. If $g > 1$, then Proposition 4.16 tells us that $HF^+(K_0, g-1) = 0$, contradicting Section 4.5. If $g = 1$, then Proposition 4.21 tells us that $\underline{HF^+}(K_0) \otimes_{\mathbb{Z}[t^{-\pm 1}]} \mathbb{Z}[t^{\pm 1}, (t-1)^{-1}] = 0$. In particular, for any $\mathbb{Z}[t^{\pm 1}]$-algebra $A$ in which $t-1$ is invertible, we see that $\underline{HF^+}(K_0) \otimes_{\mathbb{Z}[t^{\pm 1}]} A = 0$.

Taking $A = \mathbb{Q}(t)$, it follows that $\underline{HF^+}(K_0, 0) \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{Q}(t) = 0$, so $\widehat{HFK}(S^3, K, 1) \otimes \mathbb{Q}(t) = 0$ by Section 4.8. Hence $\widehat{HFK}(S^3, K, 1) \otimes \mathbb{Q} = 0$. Similarly, for any prime $p$, we can take $A = (\mathbb{Z}/p)(t)$. Thus $\widehat{HFK}(S^3, K, 1) \otimes (\mathbb{Z}/p)(t) = 0$, so $\widehat{HFK}(S^3, K, 1) \otimes (\mathbb{Z}/p) = 0$. We can conclude that $\widehat{HFK}(S^3, K, 1) = 0$, another contradiction. Therefore $g = 0$ as claimed, proving Theorem 4.3 and hence Theorem 4.2 and Theorem 4.1.

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