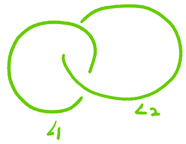
Kontsevich Invariants

Thursday, May 14, 2020 12:17 AM

Kontsevich invariants

Motivation: Gauss Linking Number

Let $L_1, L_2 : S^1 \hookrightarrow \mathbb{R}^3$ give a link



Consider the map

$$f : T^2 = S^1 \times S^1 \longrightarrow S^2$$
$$(\theta_1, \theta_2) \longmapsto \frac{L_1(\theta_1) - L_2(\theta_2)}{|L_1(\theta_1) - L_2(\theta_2)|}$$

(unit normal vector)

$f^* \omega \longleftarrow \omega \in \Omega^2(S^2)$ standard vol. form.

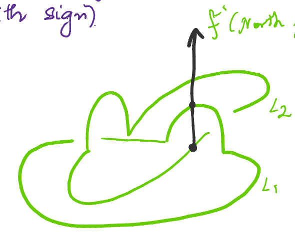
$\in \Omega^2(T^2)$

Write $lk(L_1, L_2) = \int_{T^2} f^* \omega$

Note that by Poincare duality,

$$\int_{T^2} f^* \omega = \deg(f)$$

We can also think of degree as counting preimages of any $p \in S^2$ (with sign)



$Lk(L_1, L_2) = 1$

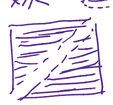
This is an example of a "configuration space integral"

Configuration spaces

M a manifold, $n \in \mathbb{N}$

$$C_n(M) : \left\{ (p_1, p_2, \dots p_n) \in M^n \;\middle|\; p_i \neq p_j \right\}$$

Ex $C_2(S^1) = S^1 \times \mathbb{R}$ 

$C_2(I) =$ 

$C_2(\mathbb{R}^3) \simeq S^2 \qquad x_1, x_2 \mapsto$ unit vector $2 \to 1$

Variations • $C_{1,1}(L_1, L_2) = S^1 \times S^1$ (1^{st} point on 1^{st} circle)

Other restrictions of subsets of points

For $lk(L_1, L_2)$ we constructed a map

$$f : \underset{(S^1 \times S^1)}{\text{Conf. space}} \longrightarrow S^2$$

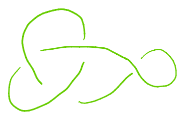
Pulled back a diff. form ω to $f^* \omega$

Then, Invariant $I(L_1, L_2) = \int_{\text{conf.}} f^* \omega$

Pulled back a

Then, Invariant $I(L_1, L_2) = \int_{\text{conf. space}} f^* \omega$

Try the same thing for knots?



$$f : C_2(S^1) \to S^2$$
$$(x_1, x_2) \mapsto \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$$

"Invariant" $= \int_{S^1 \times \mathbb{R}} f^* \omega$ "Self linking number"

Hence the above is an invariant of framed knots

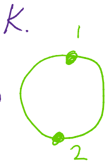
Higher dim'l invariants

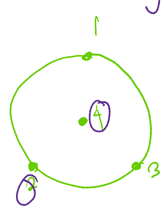
Def'n: Let $A = \{1, \dots, n\}$ $B \subseteq A$

Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a knot

$C_{A,B}(\mathbb{R}^3, K) \subseteq C_A(\mathbb{R}^3)$ such that $B \subseteq$ Knot K.

Ex: The case of the knot was $C_{2,2}(\mathbb{R}^3, K)$





$\to C_{4,3}(\mathbb{R}^3, K)$

dim = 6

3 maps $\to S^2$

Need map to S^2

For vertices $i, j \subseteq A$

$$f_{ij} : C_{A,B}(\mathbb{R}^3, K) \longrightarrow S^2$$
$$(x_1, x_2) \mapsto \frac{x_1 - x_2}{|x_1 - x_2|}$$

Ideally, we'd just integrate

$$\int_{C_{A,B}} f_{ij}^* \omega$$

But $\dim(C_{A,B}(\mathbb{R}^3, K)) = 3|A| - 2|B|$

$\deg \omega = 2.$

Instead we find a map:

$$f : C_{A,B}(\mathbb{R}^3, K) \longrightarrow \overbrace{S^2 \times S^2 \dots S^2}^{m}$$

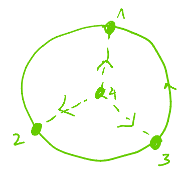
given by $f = f_{i_1 j_1} \times f_{i_2 j_2} \cdots \times f_{i_m, j_m}$

such that $m = \dfrac{3|A| - 2|B|}{2}$

then $\int_{C_{A,B}} f_{i_1j_1}^* \omega \wedge \cdots \wedge f_{i_mj_m}^* \omega$ is a number

These maps are specified by trivalent graphs

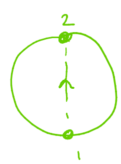
<u>Chord diagrams</u>



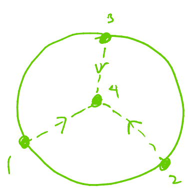
Data
- → Labelled vertices A
- → Oriented (dotted) edges
- → Specified cycle with direction
- → Knot vertices $B \subseteq A$

If (i,j) is a (dotted) edge, include $f_{ij} : C_{A,B} \to S^2$ in the wedge $\bigwedge f_{ij} \in \Omega^*(C_{A,B})$

<u>Ex</u>



Integral $I(\Gamma, K) = \int_{C_{2,2}} f_{12}^* \omega$



$I(\Gamma, K) = \int_{C_{4,3}} f_{14}^* \omega \wedge f_{24}^* \omega \wedge f_{34}^* \omega$

Given a chord diagram Γ (trivalent graph + data)

$$\boxed{I(\Gamma, K) = \int_{C_{A,B}} \bigwedge_{\{i,j\}\text{ edge}} f_{ij}^* \omega .}$$

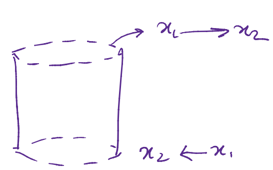
Is this an invariant?

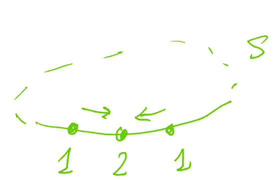
Not quite.

<u>Problems:</u>

1) Domain not compact

<u>Sol</u>: $\boxed{\text{Compactification}}$





When points collide, we need boundary faces

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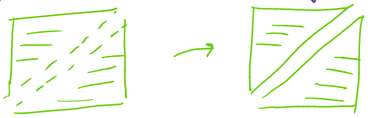
$\boxed{\begin{array}{l}\text{Add codim 1 faces when 2 points collide}\\ \text{Add codim 2 faces when 3 points collide.}\\ \quad\vdots\end{array}}$

$\overline{C_2(S^1)} =$  $\quad \partial \overline{C_2(S^1)} = S^1 \times S^0$

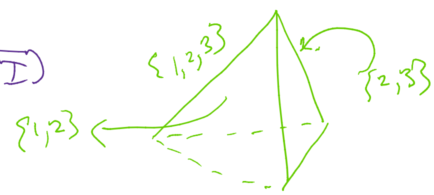
(In general, obtain a compactified conf space by attaching unit normal bundles of diagonals


)

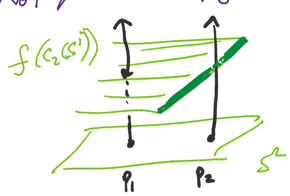
$\partial \overline{C_3(M)}$: Codim 1 faces: $\{1,2\}, \{2,3\}, \{1,3\}$

Codim 2 faces $\{1,2,3\}$

Ex $\quad \overline{C_3(I)}$



2) Not quite an isotopy invariant !



Boundary issues.

Solution: Sum up invarits that "cancel the boundary".
Then use a Stokes theorem like argument

$\boxed{\text{Weight system}}$ $w(\Gamma)$ are functions on chord diagrams Γ that satisfy

$w(\Gamma) = w(\Gamma')$ if vertex labels are interchanged.

$w(\Gamma) = -w(\Gamma')$ if an edge direction is reversed

(STU relation)

Real Invariant

$$I(K) = \frac{1}{n!} \sum_{\Gamma\text{ : chord diagrams}} I(\Gamma, K) \cdot w(\Gamma) \qquad (+ \cdots)$$

"correction term"

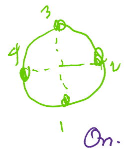
The idea is that the relations of $w(\Gamma)$ cancel ... contributions of $I(\Gamma, K)$ as we vary Γ.

diagrams

The idea is that the relations of $w(\Gamma)$ cancel out boundary contributions of $I(\Gamma, K)$ as we vary Γ.

Example of cancellation (Proof by notation)

1) via differential forms



$$\int_{C_{4,4}} f_{13}^* \wedge f_{24}^* \; - \; \int_{C_{4,3}} f_{12}^* \wedge f_{32}^* \wedge f_{42}^*$$

On $\{1,2\}$ face of $C_{4,4}$

$f_{13}^* \wedge f_{24}^*$ becomes $f_{\{1,2\},3}^* \wedge f_{\{1,2\},4}^*$

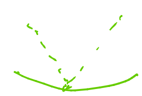
On $\{1,2\}$ face of $C_{4,3}$

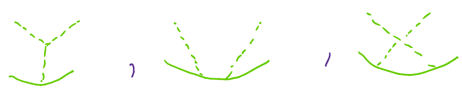
$f_{12}^* \wedge f_{32}^* \wedge f_{42}^*$ becomes $f_{12}^* \wedge f_{3\{1,2\}}^* \wedge f_{4\{1,2\}}^*$

But f_{12}^* integrates to 1 as 1 & 2 approach each other in all directions on S^2

Thus we get $f_{\{1,2\},3}^* \wedge f_{\{1,2\},4}^* - f_{3\{1,2\}}^* \wedge f_{4\{1,2\}}^* = 0$

2) Via relations for weight systems

A codim 1 face (2 points collide) that looks like  in a chord diagram

comes from 3 integrals corresponding to chord diagrams  (Same terms as STU relation)

One can check that the signs cancel out appropriately

In Summary: For knots in $\mathbb{R}^3$

Maps $f_{ij} : C_{A,B}(\mathbb{R}^3, K) \to S^2$.

Given Γ, $I(\Gamma, K) = \int_{C_{A,B}} \bigwedge f_e^* \omega$

$$I(K) = \frac{1}{n!} \sum_{\Gamma} w(\Gamma)\, I(\Gamma, K)$$

Remark 1: You can do this for knots in $\mathbb{Q}$ homology spheres.

Remark 2: Kontsevich invariants are "finite-type" invariants

Generalizations

Philosophy:
- rational homology m-sphere M
- trivalent graph Γ (labelled, oriented)

Construct $C_\Gamma(M) : \{ V(\Gamma) \hookrightarrow M \}$

$\ldots \to S^{m-1}$

Construct $C_\Gamma(M) : \{ V(\Gamma) \hookrightarrow M \}$

Construct $f : C_2(M) \to S^{m-1}$

For every edge 'e' of Γ,

$f_e : C_{2n}(M) \to C_2(M) \to S^{m-1}$

Integral $I(\Gamma, M) = \int_{C_\Gamma(M)} \bigwedge_e f_e^* \omega$

Weight system:

Let $\mathcal{G}_n$ = set of labelled trivalent graphs with $2n$ vertices

Let $A_n = \mathbb{Q}[\mathcal{G}_n] \Big/$ vertex swap, edge reversal, IHX relation

free $\mathbb{Q}$ vec. sp.

Elements of this vector space are linear combinations of

$$I(M) = \frac{1}{n!} \sum_\Gamma I(\Gamma, M)[\Gamma] \qquad (+\cdots)$$

Kontsevich Integrals on bundles

Let
$$\begin{array}{ccc} D^m & \hookrightarrow & E \\ & & \downarrow \\ & & B \end{array}$$
be a (D^m, ∂) bundle on B.

Construct a new bundle
$$\begin{array}{ccc} C_2(D^m) & \hookrightarrow & EC_2 \quad (\subseteq E \times E) \\ & & \downarrow \\ & & B \end{array}$$

On each fiber, $C_2(D^m)$, we have maps $C_2(D^m) \to S^{m-1}$ which we glue together to get $f : EC_2 \to S^{m-1}$ (like $C_2(\mathbb{R}^3) \to S^2$)

Given a trivalent graph Γ of $2n$ vertices,

We get $3n$ (# edges) maps

$$f_e : EC_{2n} \xrightarrow{\text{(restriction)}} EC_2 \xrightarrow{f} S^{m-1}$$

$$I(\Gamma, E) = \int_{EC_{2n}} \bigwedge_e f_e^* \omega$$

$$Z(E) = \sum_\Gamma I(\Gamma, E)[\Gamma] \qquad (+\cdots)$$

Z: bundles $\longrightarrow A_n$

Using this, Watanabe found elements of $\pi_k(BDiff(D^4, \partial))$

... D^4 bundles E on S^k.

Z is [illegible]

Using this, Watanabe found elements of $\pi_k(B\mathrm{Diff}(D^4,\partial))$

He did this by constructing D^4 bundles E on S^k.

$\pi : E \to S^k$ is classified by maps

$$\begin{tikzcd}
{[S^k, B\mathrm{Diff}(D^4,\partial)] = \pi_k(B\mathrm{Diff}(D^4,\partial))} \arrow[r, "Z"] & A_n
\end{tikzcd}$$

<u>Theorem</u> (Watanabe): Z is surjective and $\dim(A_n) \neq 0$ for $n = 2, 5, 9 \ldots$