How can we say 2 *knots* are *not* the same?

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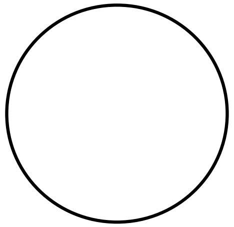
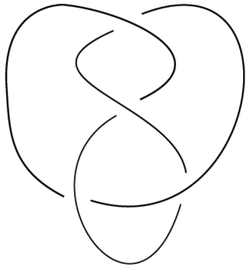
What's a knot?

A knot is a smooth embedding of the circle S^1 in $\mathbb{R}^3$. A link is a smooth embedding of the disjoint union of more than one circle

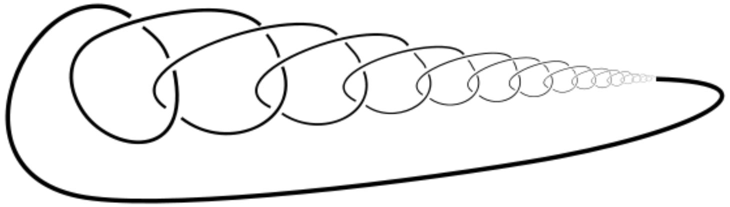
Intuitively, it's a string knotted up with ends joined up. We represent it on a plane using curves and 'crossings'.



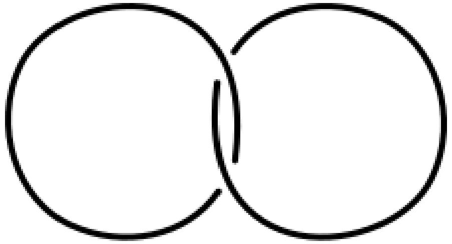
The unknot



A 'figure-8' knot



A 'wild' knot (not a knot for us)

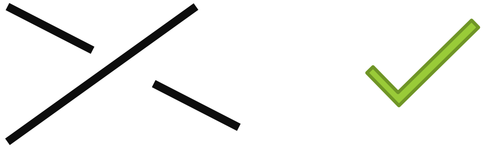


Hopf Link

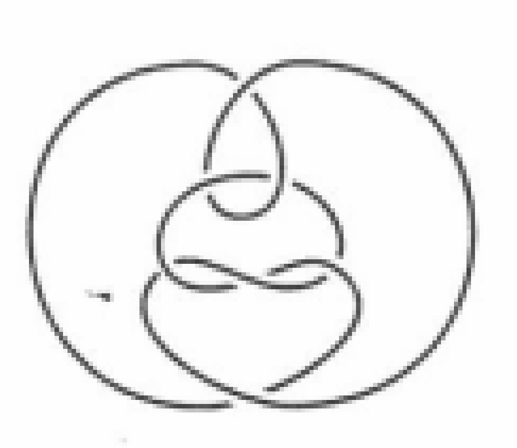
Two knots or links are the same if they have an *ambient isotopy* between them.

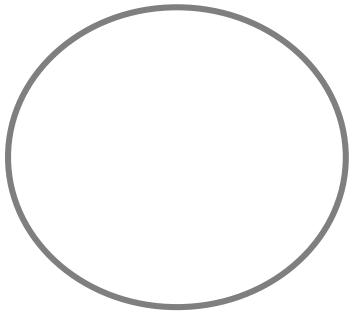
Representing a knot

Knots are represented on the plane with strands and crossings where 2 strands cross. We call this picture a knot diagram.



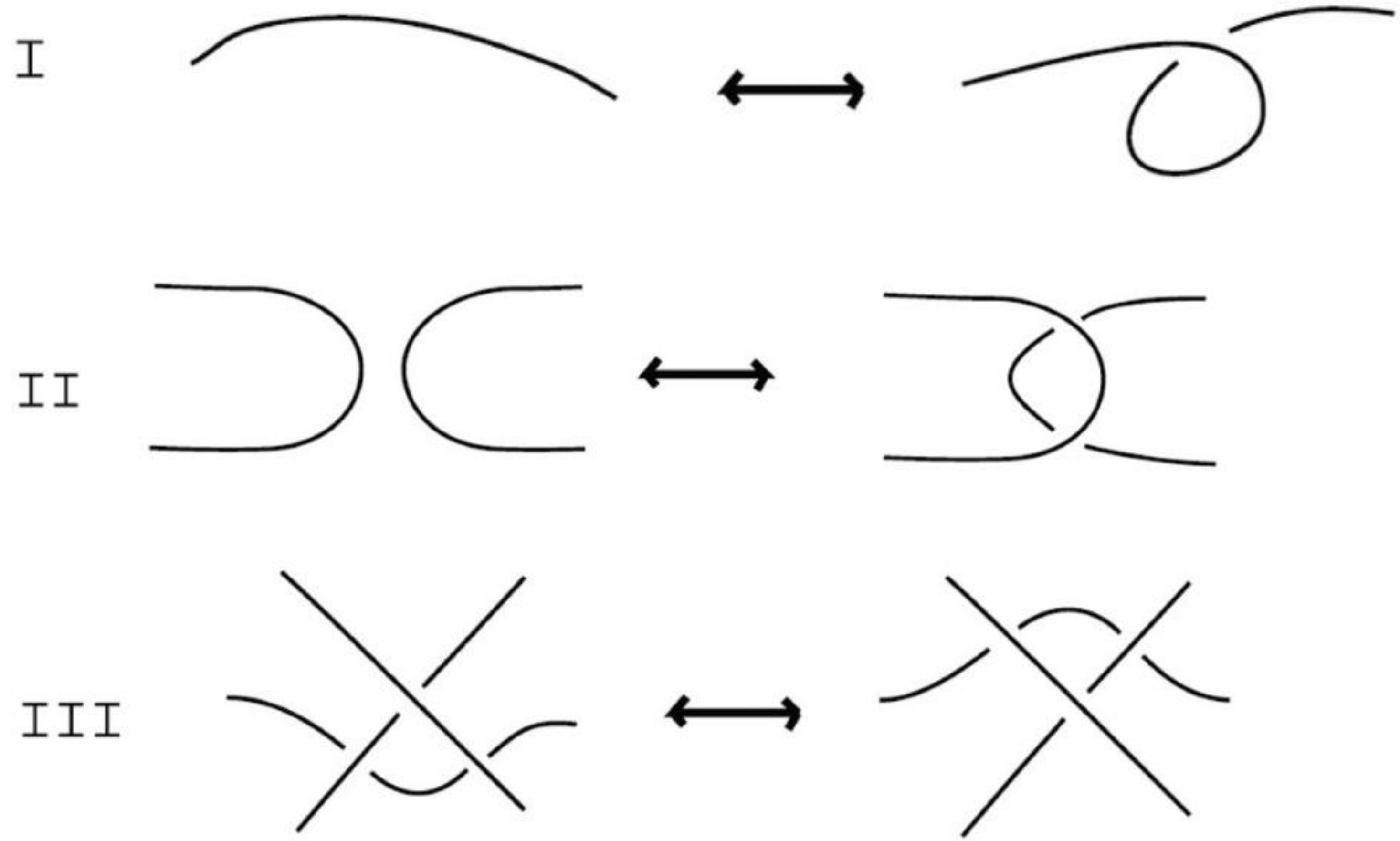
Knots can have more than one representation.



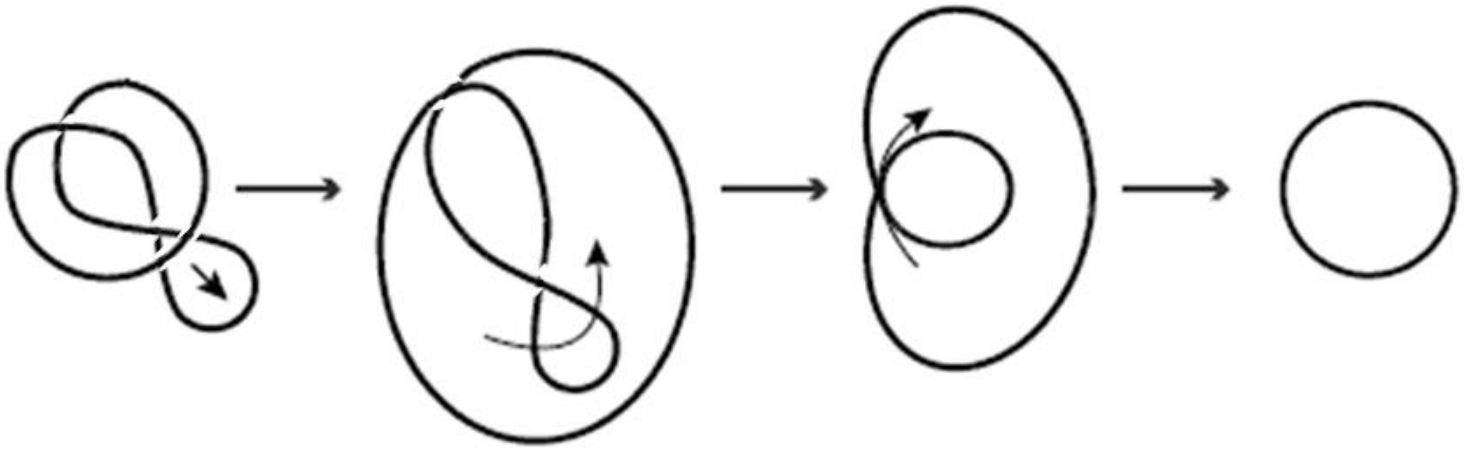


Reidemeister moves

Operations on knot diagrams that don’t change the knot or link



Reidemeister moves

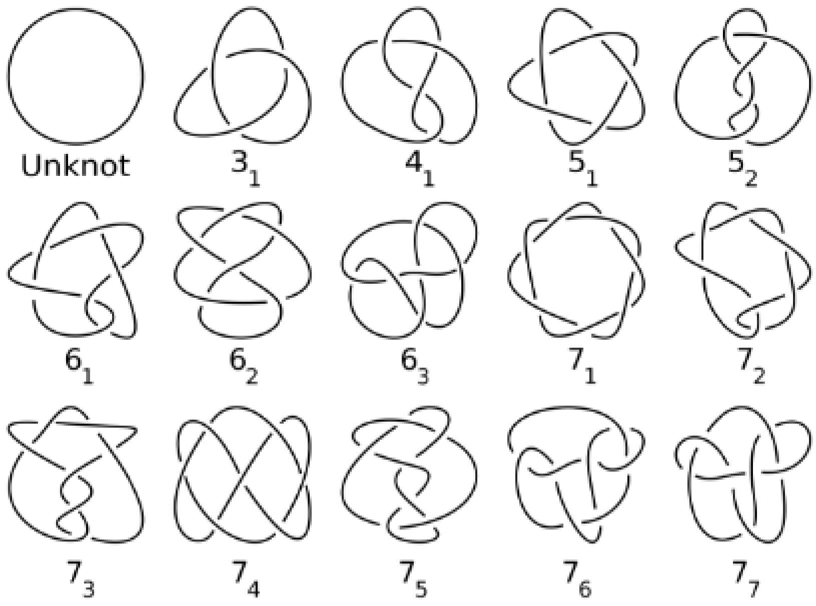


Theorem: (Reidemeister 1926) Two knot diagrams are of the same knot if and only if one can be obtained from the other through a series of Reidemeister moves.

Crossing Number

The minimum number of crossings required to represent a knot or link is called its *crossing number.*

Knots arranged by crossing number:



Knot Invariants

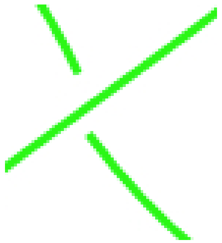
A knot/link invariant is a property of a knot/link *that is independent of representation*.

Trivial Examples:

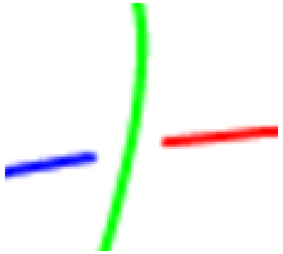
- Crossing number
- Knot Representations / ∼ where 2 representations are equivalent via Reidemester moves

Tricolorability

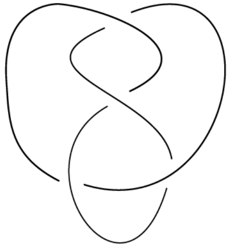
We say a knot is *tricolorable* if the strands in any projection can be colored with 3 colors such that every crossing has 1 or 3 colors and the coloring uses more than one color.

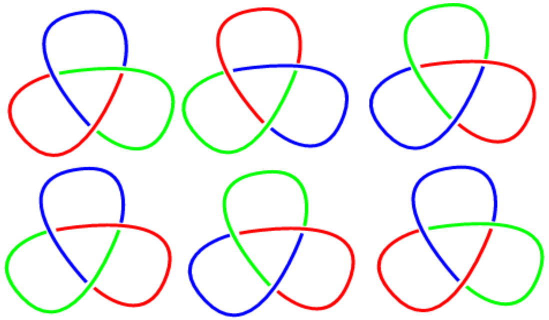


or



The unknot and the figure-8 knot are not tricolorable.



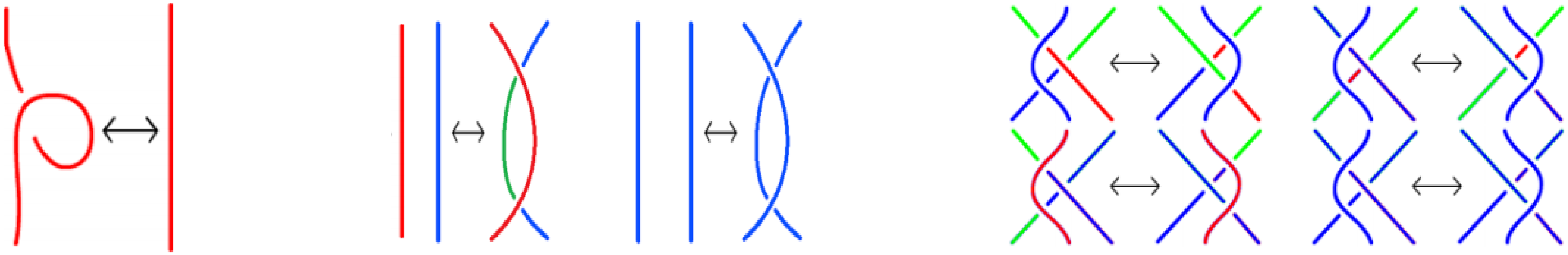


The ‘trefoil knot has 6 possible 3-colorings.

Tricolorability

Theorem: The tricolorability of a knot and the number of 3-colorings of a knot doesn't depend on it's representations. Thus, it is a knot invariant.

Proof: Tricolorability remains unchanged with Reidemeister moves.

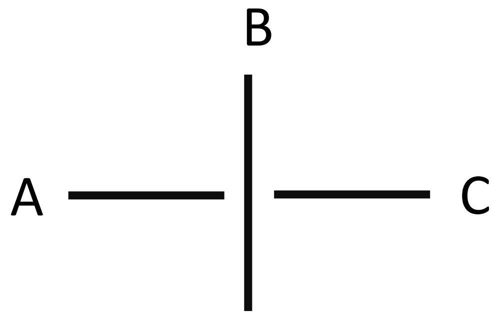


http://web.math.ucsb.edu/~padraic/ucsb_2014_15/ccs_problem_solving_w2015/Tricolorability.pdf

N - colorability

To distinguish more knots, we define an ‘n’ coloring of a knot.

You can color the strands of a knot with at most ‘n’ colors labelled 1, 2, … n such that at least 2 colors are used and at a crossing as shown, $A + B + C \equiv 0 \pmod{n}$

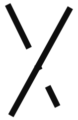


The figure-8 knot is 5-colorable.

Theorem: The n-colorability of a knot and the number of n-colorings of a knot is a knot invariant.

Jones Polynomial

The bracket polynomial <K> of a knot diagram K is a Laurent polynomial in x defined recursively as follows:

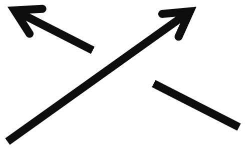
- $< O > = 1$
- $< L \coprod O > = -(x^2 + x^{-2}) < L >$
- $<$  $> = x <)(> + x^{-1} <$  $>$

Jones Polynomial

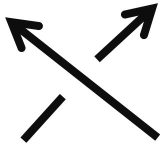
The bracket polynomial is invariant of type II and type III Reidemeister moves.

However, it multiplies a factor of $-x^3$ with each type I Reidemeister move.

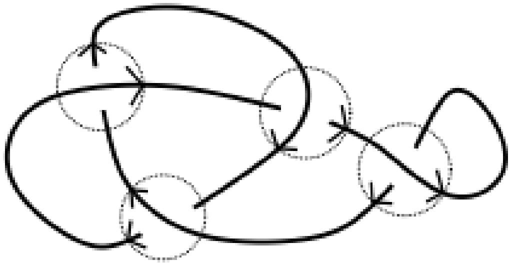
Writhe of an oriented link diagram K: w(K) is the difference between the number of positive crossings and negative crossings of the diagram.



Positive crossing



Negative crossing



Writhe number = 3 – 1 = 2

Jones Polynomial

The writhe number is independent of Type II and Type III Reidemeister moves.

Define the polynomial $P_K(t) = (-x)^{-3w(K)} < K >$. This is independent of knot diagram.

The Jones polynomial: $V_K(t) = P_K(t^{1/4})$ is hence a knot invariant.

Jones Polynomial

The Jones polynomial can distinguish chiral knots: knots that aren't the same as their mirror image. In fact, $V_{K_{flip}}(t) = V_K(t^{-1})$

(A) left-handed trefoil

(B) right-handed trefoil

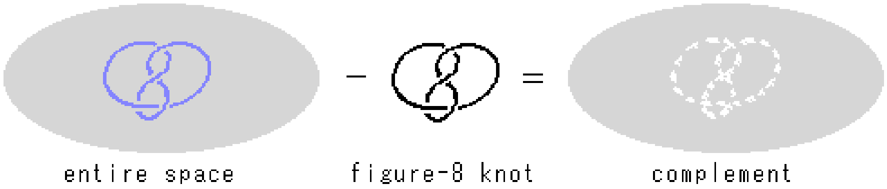
The *span* of the Jones polynomial is the difference between the highest and lowest exponent.

$$span\, V_K \leq c(K)$$

The Jones polynomial is related to the Euler characteristic of the Khovonov homology of a knot.

Knot group

The complement of a knot in $\mathbb{R}^3$ is itself an invariant. It is a topological space and hence has attributes like the fundamental group.

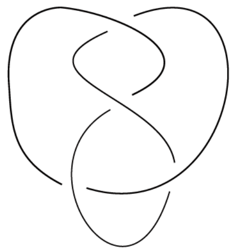


A simple algorithm by Wirtinger on the knot diagram gives a group presentation of the fundamental group of the knot complement.

Hyperbolic Volume

Some knots and links are *hyperbolic*, i.e. when embedded in S^3, the complement is a hyperbolic 3 manifold. These manifolds have finite hyperbolic volume, also an invariant.

The figure-8 knot is the simplest hyperbolic knot with hyperbolic volume 2.0298…



Hyperbolic knots are abundant: of all of the knots classified up to 16 crossings, 13 are torus knots, 20 are satellite knots and 1,701,903 are hyperbolic knots.

Total Curvature of a knot

The Fary-Milnor theorem relates the total absolute curvature of a knot embedded in Euclidean space.

Theorem: If $\oint_K |\kappa(s)|ds \leq 4\pi$ then K is the unknot.

Other knot invariants include homologies, knot polynomials etc.

References

- The “Knot Book” by Colin Adams
- http://people.virginia.edu/~btw4e/knots-2015.pdf
- http://math.uchicago.edu/~ac/alan_chang_junior_paper_spring2013.pdf