Notation Sheet

- $M \hookrightarrow \mathbb{R}^N$, a mfd w/ ∂, $p_0, p_1 \in \partial M$ $\quad v, w \in (STM)_{p_0, p_1}$
- $\mathrm{Emb}(I, M) := \left\{ \text{embeddings } f: I \to M \;\middle|\; \begin{array}{ll} f(0) = p_0 & df(0) = v \\ f(1) = p_1 & df(1) = w \end{array} \right\}$
- $C_n(M)$: Ordered Config n points in M.
- $\pi_{ij} : C_n(M) \to S^{N-1}$
  $(x_1 \cdots x_n) \mapsto \dfrac{x_i - x_j}{|x_i - x_j|}$
- $C_n\langle M\rangle$: compactification of $C_n(M)$ under $i \times (\pi_{ij})_{i<j}$
- $C_n^1\langle M\rangle$: Pull back from $(STM)^n$ (unit tangent vectors)
  $\hookrightarrow$ Points look like $((x_1, v_1), (x_2 v_2) \cdots (x_n, v_n))$
  $+ \pi_{ij}$ data if $x_i = x_j$
- $C_n^1\langle M, \partial\rangle \subseteq C_{n+2}^1\langle M\rangle$ where $x_0 = p_0$
  $x_{n+1} = p_1$
  $v_0 = v$, $v_{n+1} = w$
- $C_n^1\langle I, \partial\rangle$: Connected component Δ^n where points occur in order, +ve tangent vectors
- Stratification of $C_n\langle M, \partial\rangle$
  Indexed by $S \subsetneq \{0, \dots, n\}$
  $C_\emptyset = C_n\langle M, \partial\rangle$
  $C_S := \{(x_0, \cdots x_{n+1}) \in C_n\langle M, \partial\rangle \mid x_i = x_{i+1} \; \forall i \in S\}$
- Aligned Stratum of $C_n^1\langle M, \partial\rangle$
  $C_S^1 := \left\{ ((x_0, v_0), (x_1, v_1) \cdots (x_{n+1}, v_{n+1})) \in C_n^1\langle M, \partial\rangle \;\middle|\; \begin{array}{l} x_i = x_{i+1} \quad i \in S \\ v_i = v_{i+1} \\ = \pi_{i,i+1}(\times) \end{array} \right\}$
- $AM_n(M)$: Aligned Strata preserving maps
  $C_n^1\langle M, \partial\rangle$

$$\Delta_n = C_n'\langle I, \partial\rangle \longrightarrow C_n'\langle I, \partial\rangle$$

- $\mathrm{Emb}(I, M) \longrightarrow P_n(\mathrm{Emb}(I, M)) = AM_n(M)$ is $(n-1)(\dim M - 3)$ connected
- $E_S(M) = \mathrm{Emb}\left(I \setminus \left(\bigcup_{s \in S} J_s\right), M\right)$
- $P_n(\mathrm{Emb}(I, M)) := \operatorname{holim}_{S \subsetneq \{0, \dots, n\}} E_S(M)$

$$\underset{w.e}{\simeq} \operatorname{holim}_{S \subsetneq \{1, \dots, n\}} C'_{[n] \setminus S}$$

$$\underset{w.e}{\simeq} AM_n(M).$$

Talk: Models for $\mathrm{Emb}(I, M)$

Let M be a compact manifold with ∂.

Let $p_0, p_1 \in \partial M$ & $v, w \in (STM)_{p_0, p_1}$

$$\mathrm{Emb}(I, M) := \left\{ f : [0,1] \to M \;\middle|\; \begin{array}{ll} f(0) = p_0 & f(1) = p_1 \\ df_0 = v & df_1 = w \end{array} \right\}$$

<u>Notation</u>:

$C_n(M)$: ordered n-point config of M.

$C_n'(M)$: pullback of square

$$\begin{array}{ccc} C_n'(M) & \hookrightarrow & (STM)^n \\ \downarrow & & \downarrow \\ C_n(M) & \hookrightarrow & M^n \end{array}$$

If $f \in \mathrm{Emb}(I, M)$ f induces a map

$$ev_n(f) : C_n'(I) \longrightarrow C_n'(M)$$

$$(x_i, v_i)_i \longmapsto (f(x_i), df_{x_i}(v_i))$$

$$ev_n : \mathrm{Emb}(I, M) \longrightarrow \mathrm{Maps}(C_n'(I) \to C_n'(M))$$

Q: Does $\{ev\}_n$ contains all homotopy information of $Emb(I, M)$.

A : No.

Definition: Let $M \hookrightarrow \mathbb{R}^N$

- $A_n\langle M\rangle := M^n \times (S^{N-1}_{ij})^{\binom{n}{2}}_{i<j\le n}$
- $\pi_{ij} : C_n(M) \longrightarrow S^{N-1}_{ij}$
$$(x_1, \dots x_n) \mapsto \frac{(x_i - x_j)}{|x_i - x_j|}$$
- $C_n\langle M\rangle$ is the compactification of $C_n(M)$ in $A_n\langle M\rangle$ by taking its closure under $i \times (\pi_{ij})_{i<j}$

Exercise: Show that proj of $C_n\langle M\rangle \longrightarrow M^n$ is surjective, continuous.

Exercise* / Prop: $C_n\langle M\rangle \simeq C_n(M)$ ....
Work out for $C_3(I)$, $C_2(S^n)$.
"Proof" : Find a ngo $C_n[M] \to C_n\langle M\rangle$

Define $C'_n\langle M\rangle$ as pullback
$$\begin{tikzcd} C'_n\langle M\rangle \arrow[r] \arrow[d] & (STM)^n \arrow[d] \\ C_n\langle M\rangle \arrow[r] & M^n \end{tikzcd}$$

Think of $C'_n\langle M\rangle$ as contains
$((x_1, v_1) \dots (x_n, v_n))$ if $x_i = x_j$ add. π_{ij} info.

$f \in Emb(I, M)$, induces $ev_n^{(f)} : C'_n\langle I\rangle \to C'_n\langle M\rangle$

Q: Does $\{ev_n\}$ contain all htpy info of $Emb(I, M)$.

Short Answer : YES

Reference : Dev Sinha : Topology of spaces of knots: cosimplicial models.

Answer : Let dim M > 3 $\quad AM_n(M)$

Longer Answer:

$$ev_n : Emb(I,M) \longrightarrow \left\{ \begin{array}{l} \text{aligned strata preserving} \\ Map(C_n'\langle I, \partial\rangle \to C_n'\langle M, \partial\rangle) \end{array} \right\}$$

is $(n-1)(\dim M - 3)$ connected.

Rmk 1: As n increases, ev_n is more and more connected. Taking inverse limit, $AM_\infty(M)$ is w.e $Emb(I,M)$.

Rmk2: Connectivity estimates come from Embedding Calculus methods of Goodwillie, Klein, Weiss ....

Rmk3: $\dim M = 3$. Goodwillie calc estimates don't work. Guess $(n-1)(\dim M - 3) = 0$. 0 connected iso on π_0..? $\pi_0 Emb(I,M) \Longleftrightarrow$ knots.

Thm (Budney, Sinha, Conant Koytcheff '14, Kosanovic '19 PhD Thesis)
Emb calc invts are finite type

$$\pi_0(AM_\infty(Emb(I,M)) \longleftrightarrow \left\{ \begin{array}{l} \text{eq. classes of knots} \\ \text{distinguished by} \\ \text{finite type invts} \end{array} \right\}$$

$(\dim M = 3)$

$\updownarrow$ BIG OPEN QUESTION?

$\pi_0(Emb(I,M))$

Mapping Space Models : $AM_n(M)$.

$C_n\langle M\rangle$ is a stratified space

Strata C_S are indexed by subsets $S \subseteq \{1, \cdots n\}$.

Strata inclusion $C_S \subset C_{S'}$ induced by $S' \subseteq S$.

$\Updownarrow$

doubling maps $C_{n-\#S}\langle M\rangle \to C_{n-\#S'}\langle M\rangle$

Strata of $C_n\langle M\rangle$

$C_\emptyset$: $C_n\langle M\rangle$

$\{ \; | \; i \in S \quad x_i = x_{i+1} \}$

$C_S : \{(x_1, \dots x_n) \in C_n\langle M\rangle \mid$ where we say $x_i = x_{i+1}$

Note : Abuse of notation under $C_n\langle M\rangle \to M^n$.

Ex : $C_{\{1,3\}} \hookrightarrow C_\emptyset = C_4\langle M\rangle$

$$\begin{gathered} C_{\{1,3\}} \\ \text{\rotatebox{90}{=}} \\ \{(x_0, x_1, x_3, x_3)\} \end{gathered}$$

Fixing boundary conditions.

Define $C_n\langle M, \partial\rangle \subseteq C_{n+1}\langle M\rangle$ as the subspace where

$(x_0, \dots\dots x_{n+1})$

$x_0 = p_0$
$x_{n+1} = p_1$

Strata of $C_n\langle M, \partial\rangle$ are proper subsets $S \subsetneq \{0, \dots n\}$

$C_S := \{x_i = x_{i+1} \quad i \in S.\}$

Denote $C_n\langle I, \partial\rangle$ as the connected comp. where points occur in order

Lemma / Exercise : $C_n\langle I, \partial\rangle \cong \Delta^n$ std n-simp

Strata are faces.

Tangent Vectors

$C_n'\langle M, \partial\rangle \subset C_{n+2}'\langle M, \partial\rangle$ is the subspace

$(x_i, v_i)_{0 \le i \le n+1}$

$x_0 = p_0 \qquad v_0 = v$
$x_{n+1} = p_1 \qquad v_{n+1} = w.$

Lemma : Let $f \in \mathrm{Emb}(I, M)$, $ev_n(f) : C_n'\langle I, \partial\rangle \to C_n'\langle M, \partial\rangle$

If x_i & x_j "collide" in $C_n'\langle I, \partial\rangle$, then

$f(x_i) = f(x_j) \qquad \pi_{ij}(\cancel{x}) = f(v_i) = f(v_j)$

Aligned Strata of $C_n'\langle M, \partial\rangle$.

$$C_S' := \left\{ ((x_0,v_0),\dots,(x_{m+1},v_{m+1})) \;\middle|\; \begin{array}{l} \text{If } x_i = x_j \\ \pi_{ij} = v_i = v_j \end{array} \right\}$$

$C_n'\langle I,\partial\rangle :=$ connected comp^n of Δ with true tang. vector.

Define
$$AM_n(M) := \left\{ \begin{array}{l} \text{Maps: } \Delta^n \to C_n'\langle M,\partial\rangle \\ \text{that are strata preserving} \\ \text{and whose image lies in} \\ \text{aligned strata.} \end{array} \right\}$$

<u>Exercise</u>: $AM_1(M) \simeq \mathrm{Imm}(I, M)$.

<u>Applications</u>: 1) New Perspectives in Self linking: $AM_3(\mathbb{R}^3)$

2) Embedding Calc. invariants are finite type.

<u>Goodwillie calculus</u>

<u>General idea</u>: N, M cpt mfds $\dim M - \dim N \geq 3$

Approximate $\mathrm{Emb}(N,M)$ by $P_n(\mathrm{Emb}(N,M))$ for $n = 0,1,2,\dots$ to get a tower of fibrations

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_n(\mathrm{Emb}(N,M)) \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_1\,\mathrm{Emb}(N,M) \arrow[d] \\
\mathrm{Emb}(N,M) \arrow[r] \arrow[ur] \arrow[uuur] & P_0(\mathrm{Emb}(N,M))
\end{tikzcd}$$

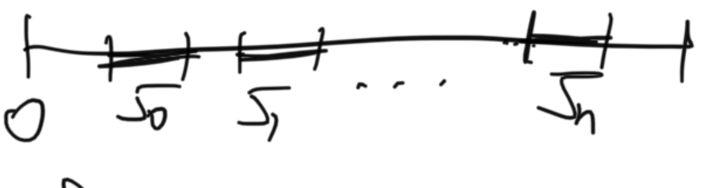
Such that $\mathrm{Emb}(N,M) \to P_n(\mathrm{Emb}(N,M))$ are $(n-1)(\dim M - \dim N - 2) + 1 - \dim N$ connected.

Take inverse limit
$$\mathrm{Emb}(N,M) \overset{w.e}{\simeq} P_\infty \dots$$

What are $P_n(\mathrm{Emb}(I,M)) \overset{?}{\longleftrightarrow} AM_n(M)$

Goodwillie Punctured Knots Model.

$I = [0,1]$. Take $n+1$ disjoint intervals $J_0, \dots\dots J_n$ in that order, not containing 0 or 1.



Let $S \subseteq \{0, \dots n\}$

If $S \subseteq S'$, restriction maps.

$$f_{S \subseteq S'} : \mathrm{Emb}\Big(I \setminus \Big(\bigcup_{s \in S} J_s\Big), M\Big) \to \mathrm{Emb}\Big(I \setminus \bigcup_{s' \in S'} J_{s'}, M\Big)$$

fits into a diagram of a $(n+1)$ cube.

$$\begin{tikzcd}[column sep=small]
 & & J_1, J_2) \arrow[rr] & & \mathrm{Emb}(I \setminus (J_0 \cup J_1, J_2), M) \\
\mathrm{Emb}(I \setminus J_1, M) \arrow[urr] \arrow[rr] & & \mathrm{Emb}(I \setminus (J_0 \cup J_1), M) \arrow[urr] & & \\
 & & & & J_0, J_2 \arrow[uu] \\
 & \mathrm{Emb}(I \setminus J_2, M) \arrow[uuu, dashed] \arrow[r, dashed] & \arrow[uu] \arrow[rr] & & \\
\mathrm{Emb}(I, M) \arrow[uuu] \arrow[ur] \arrow[rr] & & \mathrm{Emb}(I \setminus J_0, M) \arrow[uurr] & &
\end{tikzcd}$$

$S = \emptyset$

Consider the "punctured" diagram : without $\mathrm{Emb}(I, M)$.

$$P_n(\mathrm{Emb}(I, M)) := \underset{S \subsetneq [n]}{\mathrm{holim}} \; \mathrm{Emb}\Big(I \setminus \bigcup_{s \in S} J_s, M\Big).$$

$\exists$ maps $\mathrm{Emb}(I, M) \to P_n(\mathrm{Emb}(I, M))$ that are $(n-1)(\dim M - 3)$ conn : Goodwillie.

Homotopy limits of Diagrams

Let D be a small category. (Diagram)

Let $F : D \to \mathrm{Top}$

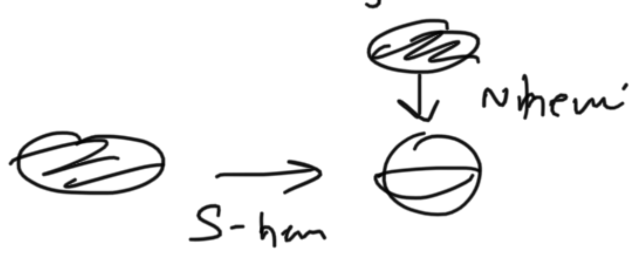
Let F : D → ...

Lim F is a space with a cone to F(D)

Such that any cone (X, C) to D comutes with a map $X \to \lim F$.

<u>Example</u>: D:
$$\begin{array}{ccc} & & B \\ & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

F_1 :



$\lim F_1$:

$\lim F$: $\{ (a,b) \in A \times B \mid f(a) = g(b) \}$

F_2 :

$\lim F_2$: $\emptyset$

Notice $F_1 \simeq F_2$ h.e

but $\lim F_1 \not\simeq F_2$.

<u>Rmk</u>: Limits and homotopy eq. don't commute.

<u>Homotopy Limits</u>

<u>Fact</u>: Homotopy limits comute with h.e & w.e.

<u>Definition by example 1</u> : $F(D)$: $A \xrightarrow{f} B$

holim F : $\{ (a, b, \underset{\text{Path}}{[0,1] \to B} \text{ joining } f(a) \,\&\, b) \}$

$\lim F$: $\{ (a,b) \mid f(a) = b \}$

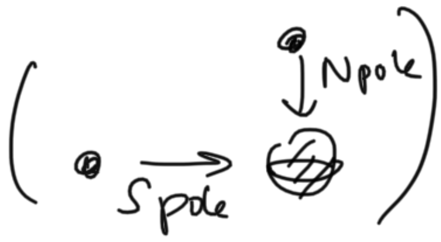
B

Example 2 : $F(D)$: $A \xrightarrow{f} C \xleftarrow{g}$

holim F : $\{(a, b, c, \text{ Path } f(a) \text{ to } c, \text{ Path } g(b) \text{ to } c)\}$

$\wr$

$\{(a, b, \text{ Path } f(a) \text{ to } g(b))\}$

holim  $\simeq \Omega S^2$ based loop space

holim  $\simeq D^2 \times D^2 \times \Omega S^2 \simeq \Omega S^2$

Example 3

$$F(D) : \begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "f\circ g"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

holim F : $\{(a, b, c, \text{ Path } P_1\ f(a) \to b, \text{ Path } P_2\ g(b) \text{ to } c,$
$\text{Path } P_3\ f(g(a)) \text{ to } c, \text{ triangle in whose } \partial = g(P_1) \cup P_2 \cup P_3)\}$

Idea: n- composible morphisms $\xrightarrow{\text{add}}$ n-simplex

Actual Definition

$F : D \to \mathrm{Top}$

Def. $D\downarrow\_ : D \to \mathrm{Cat}$ " over category"
$d \mapsto D\downarrow d$

Def $|D\downarrow -| : D \to \mathrm{Top}$
$d \mapsto |D\downarrow d|$

$$\operatorname{holim} F := \mathrm{Nat}(|D\downarrow -|, F)$$

Back to Embedding Calculus

Let $E_S(M) = \mathrm{Emb}(I \setminus \bigcup_{s \in S} J_s, M)$

Punctured cube

$$\begin{tikzcd}
 & E_{\{1,2\}} \arrow[r] & E_{\{0,1,2\}}(M) \\
E_{\{1\}}(M) \arrow[ur] & \arrow[r] & E_{\{0,1\}}(M) \arrow[u] \\
 & E_{\{2\}}(M) \arrow[uu] \arrow[r] & E_{\{0\}}(M) \arrow[u]
\end{tikzcd}$$

Exercise: $E_S(M) \simeq C^1_{\#S-1}(M, \partial)$

Idea Replace cube with $C^1_{\#S-1}(M)$. but

There is no way to define $f_{S \subseteq S'}$.

Replace $E_S(M)$ by $C^1_{\#S-1}\langle M, \partial \rangle$

$f_{S \subseteq S'}$ given by strata inclusion

$$C^1_{[n] \setminus S} \hookrightarrow C^1_{[n] - S'}$$